Loop Erased Walks and Uniform Spanning Trees

Martin T. Barlow*

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1 Introduction

The uniform spanning tree has had a fruitful history in probability theory. Most notably, it was the study of the scaling limit of the UST that led Oded Schramm [Sch00] to introduce the SLE process, work which has revolutionised the study of two dimensional models in statistical physics. But in addition, the UST relates in an intrinsic fashion with another model, the *loop erased random walk* (or LEW), and the connections between these two processes allow each to be used as an aid to the study of the other.

These notes give an introduction to the UST, mainly in $\mathbb{Z}^d$. The later sections concentrate on the UST in $\mathbb{Z}^2$, and study the relation between the intrinsic geometry of the UST and Euclidean distance. As an application, we study random walk on the UST, and calculate its asymptotic return probabilities.

This survey paper contains many results from the papers [Lyo98, BLPS, BKPS04], not always attributed.

Finite graphs. A graph G is a pair $G = (V, E)$. Here V is the set of vertices (finite or countably infinite) and E is the set of edges. Each edge e is a two element subset of V – so we can write $e = \{x, y\}$. We think of the vertices as points, and the edges as lines connecting the points. It will sometimes be useful to allow multiple edges between points. If $\{x, y\} \in E$ we write $x \sim y$, and say that x, y are neighbours, and that x is connected to y.

Now for some general definitions.

(1) We define $d(x, y)$ to be the length n of the shortest path $x = x_0, x_1, \ldots, x_n = y$ with $x_{i-1} \sim x_i$ for $1 \le i \le n$. If there is no such path then we set $d(x, y) = \infty$.