APPENDIX A

Radon transform and propagation of singularities in $\mathbf{R}^n$

In Theorem 5.2 of Chap. 4, we proved the singularity expansion of the Radon transform for an asymptotically hyperbolic metric using the parametrics for the perturbed wave equation. It is also the case for the wave equation in the asymptotically Euclidean space. In this appendix, we state the precise results as well as the relations between the Radon transform, the asymptotic profiles of the wave equation and scattering matrices in a general short-range perturbation regime. The main results are Theorem 1.14, Lemma 1.17, which can be utilized directly in the inverse scattering for the wave equation, and Theorems 6.7, 6.10, which show how the Radon transform is related with the propagation of singularities.

The Radon transform associated with the Euclidean metric is defined by

$$\left(\mathcal{R}_0 f\right)(s,\theta) = \int_{s=x\cdot\theta} f(x)d\Pi_x, \quad s\in\mathbf{R}, \quad \theta\in S^{n-1},$$

$d\Pi_x$ being the measure induced on the hyperplane $\{x\in\mathbf{R}^n;\, s = x\cdot\theta\}$ from the Lebesgue measure dx on $\mathbf{R}^n$. This is rewritten as

$$\left(\mathcal{R}_0 f\right)(s,\theta) = (2\pi)^{(n-1)/2}\int_{-\infty}^{\infty} e^{isk}\widehat{f}(k\theta)dk,$$

where $\widehat{f}$ is the Fourier transform:

$$\widehat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} e^{-ix\cdot\xi}f(x)dx.$$

Let us consider the Riemannian metric on $\mathbf{R}^n$ satisfying the following condition:

$$|\partial_x^\alpha(g_{ij}(x)-\delta_{ij})| \leq C_\alpha(1+|x|)^{-1-\epsilon_0-|\alpha|}, \quad \forall\alpha, \tag{0.1}$$

where $\epsilon_0>0$ is a constant. In Chap. 2, §7, we have already constructed a generalized Fourier transformation $\mathcal{F}^{(\pm)}$ for Δ_g. As in Chap. 2, §7, we construct $\mathcal{F}_\pm$ from $\mathcal{F}^{(\pm)}$, and define the modified Radon transform $\mathcal{R}_\pm$ by

$$\mathcal{R}_\pm f(s,\theta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isk}(\mathcal{F}_\pm f)(k,\theta)dk.$$

For the Euclidean Laplacian in $\mathbf{R}^n$ this turns out to be

$$\mathcal{R}_\pm = \left(\mp\partial_s + 0\right)^{\frac{n-1}{2}}\mathcal{R}_0.$$

The main issue of this chapter is the *singular support theorem* for $\mathcal{R}_\pm$. We construct $\varphi(x,\theta)\in C^\infty(\mathbf{R}^n\times S^{n-1})$ such that

$$|\partial_\theta^\alpha\partial_x^\beta(\varphi(x,\theta)-x\cdot\theta)| \leq C_{\alpha\beta}(1+|x|)^{-|\beta|-\epsilon_0},$$