CHAPTER 6

Boundary control method

1. Brief introduction to the boundary control method

1.1. Wave equation and Gel'fand inverse problem. Let $\mathcal{N}$ be an n-dimensional complete connected Riemannian manifold with boundary $\partial\mathcal{N}$. We shall consider an IBVP (initial-boundary value problem) for the wave equation

$$\partial_t^2 u = \Delta_g u \quad \text{on} \quad \mathcal{N} \times (0, \infty),$$

where Δ_g is the Laplace-Beltrami operator. In local coordinates

$$\Delta_g = g^{-1/2}\partial_i(g^{ij}g^{1/2}\partial_j), \quad g = \det(g_{ij}).$$

We impose the initial condition

$$u\big|_{t=0} = \partial_t u\big|_{t=0} = 0,$$

and the boundary condition

$$\partial_\nu u\big|_{\partial\mathcal{N}\times(0,\infty)} = f \in C_0^\infty(\partial\mathcal{N} \times (0, \infty)).$$

Here ν is the outer unit normal to $\partial\mathcal{N}$. Let $u^f(x,t)$ be the solution to the above IBVP. We measure u^f on $\partial\mathcal{N} \times (0,\infty)$, and call

$$\Lambda^h : f \to u^f\big|_{\partial\mathcal{N}\times(0,\infty)} \tag{1.1}$$

a *hyperbolic Neumann-to-Dirichlet map.* The basic question we address is the following one.

Question Assume we know Λ^h. Can we determine $(\mathcal{N}, g)$, i.e. the manifold $\mathcal{N}$ and the metric g?

This is the *Gel'fand inverse problem* (stated in a slightly different form, [**37**]). Note that Λ^h is an operator defined on $\partial\mathcal{N} \times (0,\infty)$. Starting from the knowledge on $\partial\mathcal{N} \times (0,\infty)$, the first issue is the topology of $\mathcal{N}$, and the second issue is the Riemannian structure.

The answer to the above question is affirmative when $\mathcal{N}$ is compact, and also for non-compact $\mathcal{N}$ with some additional geometric assumption. To fix the idea, in this chapter, $\mathcal{N}$ means either any compact connected Riemannian manifold with boundary, or when dealing with the non-compact case, the manifold Ω^c discussed in Chap. 5, §4. However, the arguments given below also work for non-compact manifolds possesing the spectral representation as in the case of Ω^c. Note that in both cases $\partial\mathcal{N}$ is compact.