CHAPTER 5

# Introduction to inverse scattering

Suppose we are given two asymptotically hyperbolic metrics which differ only on a compact set. If the associated scattering operators coincide, one can show that these two metrics coincide up to a diffeomorphism. This result can be extended to manifolds with asymptotically hyperbolic ends when two metrics coincide on one end having a regular infinity. The aim of this chapter is to explain the idea of the proof of these theorems.

## 1. Local problem on $\mathbf{H}^n$

Recall that in the geodesic polar coordinates centered at $(0,1)$, the metric on $\mathbf{H}^n$ takes the form

$$ds^2 = (dr)^2 + \sinh^2 r\,(d\theta)^2,$$

where $(d\theta)^2$ is the standard metric on $S^{n-1}$ (see formula (1.4) in Chap. 1). Letting $y = 2e^{-r}$ and $x = \theta$, one can rewrite the above metric as

$$ds^2 = \left(\frac{dy}{y}\right)^2 + \left(\frac{1}{y} - \frac{y}{4}\right)^2 (dx)^2, \quad y \in (0,2].$$

Suppose this metric is perturbed so that

$$ds^2 = \frac{(dy)^2 + (dx)^2 + A(x,y,dx,dy)}{y^2},$$

with $A(x,y,dx,dy)$ satisfying the assumption (A-4) of Chap. 3, §3. The theorem we are going to prove is as follows.

**Theorem** 1.1. *Suppose we are given two Riemannian metrics $G^{(p)}$, $p = 1,2$, on $\mathbf{H}^n$ satisfying the above assumption. Suppose their scattering operators coincide. Suppose furthermore $G^{(1)}$ and $G^{(2)}$ coincide except for a compact set. Then $G^{(1)}$ and $G^{(2)}$ are isometric.*

The proof is done by the following steps. Let $B_a \subset \mathbf{H}^n$ be a ball of radius $a$ with respect to the unperturbed metric centered at $(0,1)$ such that $G^{(1)} = G^{(2)}$ outside $B_a$. We first take a geodesic sphere $S_a = \partial B_a$, and consider the boundary value problem for the Laplace-Beltrami operators in the interior domain $B_a$. Then the associated Dirichlet-to-Neumann map (or Neumann-to-Dirichlet map) coincide. We use the boundary control method of Belishev-Kurylev to show that $G^{(1)}$ and $G^{(2)}$ are isometric in $B_a$ (see [**10**] and [**77**]).

## 2. Scattering operator and N-D map

**2.1. Restriction of the generalized eigenfunctions to a surface.** Let us start with preparing local regularity estimates for the resolvent $R(k^2 \pm i0)$ constructed in Chap. 2. We first introduce some notation in $\mathbf{R}^n$. Letting $\widehat{f}(\xi)$ be the