CHAPTER 4

Radon transform and propagation of singularities in $\mathbf{H}^n$

The purpose of this chapter is to extend Theorem 1.6.6 to the asymptotically hyperbolic metric on $\mathbf{R}^n_+$ in the sense of singularity expansion.

1. Geodesic coordinates near infinity

1.1. Geodesic coordinates. We shall study the metric

$$ds^2 = y^{-2}\Big((dx)^2 + (dy)^2 + A(x,y,dx,dy)\Big) \tag{1.1}$$

on $\mathbf{R}^n_+$ defined in Chapter 2, Subsection 2.1, i.e. the metric satisfying the condition (C) in Chap. 2. Our aim is to transform (1.1) into the following canonical form

$$ds^2 = y^{-2}\Big((dx)^2 + (dy)^2 + B(x,y,dx)\Big) \tag{1.2}$$

in the region $0 < y < y_0$, y_0 being a sufficiently small constant, where $B(x,y,dx)$ is a symmetric covariant tensor of the form

$$B(x,y,dx) = \sum_{i,j=1}^{n-1} b_{ij}(x,y)dx^i dx^j.$$

Passing to the variable $z = \log y$, we rewrite the Laplace-Beltrami operator Δ_g associated with (1.1) as

$$\begin{aligned}\Delta_g = \partial_z^2 + e^{2z}\partial_x^2 + \sum_{i,j=1}^{n-1} a^{ij}(x,e^z)e^{2z}\partial_{x_i}\partial_{x_j} \\ + 2\sum_{i=1}^{n-1} a^{in}(x,e^z)e^z\partial_{x_i}\partial_z + a^{nn}(x,e^z)\partial_z^2\end{aligned}$$

up to 1st order terms. Then $\big(g^{ij}\big)$ in the variables x and z takes the form

$$g^{ij} = \begin{cases} e^{2z}\big(\delta^{ij} + h^{ij}(x,z)\big), & 1 \le i,j \le n-1, \\ e^z h^{in}(x,z), & 1 \le i \le n-1, \\ 1 + h^{nn}(x,z), & i,j = n, \end{cases} \tag{1.3}$$

where $h^{ij}(x,z)$ satisfies in the region $z < 0$

$$|\partial_x^\alpha \partial_z^\beta h^{ij}(x,z)| \le C_{\alpha\beta} W(x,z)^{-\min(|\alpha|+\beta,1)-1-\epsilon_0}, \tag{1.4}$$

and

$$W(x,z) = 1 + |z| + \log\big(|x|+1\big).$$