CHAPTER 3

Manifolds with hyperbolic ends

1. Classification of 2-dimensional hyperbolic manifolds

The hyperbolic manifold is, by definition, a complete Riemannian manifold with all sectional curvatures equal to -1. General hyperbolic manifolds are constructed by the action of discrete groups on the upper-half space. The resulting quotient manifold is either compact, or non-compact but of finte volume, or non-compact with infinite volume. In the latter two cases, the manifold can be split into bounded part and unbounded part, this latter being called the end. To study the general structure of ends is beyond our scope. We briefly look at the 2-dimensional case.

1.1. Möbius transformation. Recall that $\mathbf{C}_+ = \{z = x + iy\,; y > 0\}$ is a 2-dimensional hyperbolic space equipped with the metric

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}. \tag{1.1}$$

Let $\partial\mathbf{C}_+ = \partial\mathbf{H}^2 = \{(x,0)\,; x \in \mathbf{R}\} \cup \infty = \mathbf{R} \cup \infty$. For a matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R})$$

the Möbius transformation is defined by

$$\mathbf{C}_+ \ni z \to \gamma \cdot z := \frac{az+b}{cz+d}, \tag{1.2}$$

which is an isometry on $\mathbf{H}^2$. Since γ and $-\gamma$ define the same action, one usually identifies them and considers the factor group:

$$PSL(2,\mathbf{R}) := SL(2,\mathbf{R})/\{\pm I\}.$$

The non-trivial Möbius transformations γ are classified into 3 categories :

$$\begin{aligned}
\textit{elliptic} &\Longleftrightarrow \text{there is only one fixed point in } \mathbf{C}_+ \\
&\Longleftrightarrow |\mathrm{tr}\,\gamma| < 2, \\
\textit{parabolic} &\Longleftrightarrow \text{there is only one degenerate fixed point on } \partial\mathbf{C}_+ \\
&\Longleftrightarrow |\mathrm{tr}\,\gamma| = 2, \\
\textit{hyperbolic} &\Longleftrightarrow \text{there are two fixed points on } \partial\mathbf{C}_+ \\
&\Longleftrightarrow |\mathrm{tr}\,\gamma| > 2.
\end{aligned}$$

1.2. Fuchsian group. Let Γ be a discrete subgroup of $SL(2,\mathbf{R})$, which is usually called a *Fuchsian* group. As a short introduction to the theory of Fuchsian groups, we refer [**81**]. Let $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$ be the fundamental domain by the action (1.2). Γ is said to be *geometrically finite* if $\mathcal{M}$ is chosen to be a finite-sided convex