CHAPTER 2

Perturbation of the metric

We shall study in this chapter spectral properties of $-\Delta_g$, where Δ_g is the Laplace-Beltrami operator associated with a Riemannian metric, which is a perturbation of the hyperbolic metric on $\mathbf{H}^n$. We shall prove the limiting absorption principle, construct the generalized Fourier transform and introduce the scattering matrix. To study $\mathbf{H}^n$ in an invariant manner, it is better to employ the ball model and geodesic polar coordinates centered at the origin. However, we use the upper-half space model, since it is of independent interest, necessary in order to make the arguments in Chapter 1 complete by the method adopted here, and also of a preparatory character to deal with hyperbolic ends in Chapter 3.

1. Preliminaries from elliptic partial differential equations

1.1. Regularity theorem. In this section, for the notational convenience, we denote points $x \in \mathbf{R}^n$ by $x = (x_1, \cdots, x_n)$. We consider the differential operator

$$A = \sum_{|\alpha| \leq 2} a_\alpha(x)(-i\partial_x)^\alpha$$

defined on $\mathbf{R}^n$. The coefficients $a_\alpha(x)$ are assumed to satisfy

$$a_\alpha(x) \in C^\infty(\mathbf{R}^n), \quad \partial_x^\beta a_\alpha(x) \in L^\infty(\mathbf{R}^n), \quad \forall\beta,$$

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq C|\xi|^2, \quad \forall x \in \mathbf{R}^n, \quad \forall \xi \in \mathbf{R}^n,$$

C being a positive constant. A function $u \in L^2_{loc}(\mathbf{R}^n)$ is said to be a weak solution of $Au = f$ if it satisfies

$$\int_{\mathbf{R}^n} u(x)\overline{A^\dagger\varphi(x)}dx = \int_{\mathbf{R}^n} f(x)\overline{\varphi(x)}dx, \quad \forall\varphi \in C_0^\infty(\mathbf{R}^n),$$

where $A^\dagger$ is the formal adjoint of A.

Theorem 1.1. *If $u \in L^2(\mathbf{R}^n)$ is a weak solution of $Au = f$ and $f \in H^m(\mathbf{R}^n)$ for some $m \geq 0$, then $u \in H^{m+2}(\mathbf{R}^n)$, and*

$$\|u\|_{H^{m+2}(\mathbf{R}^n)} \leq C(\|u\|_{L^2(\mathbf{R}^n)} + \|f\|_{H^m(\mathbf{R}^n)}).$$

For the proof see e.g. [**101**]. By using Theorem 1.1, one can prove the following inequality. Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary, and Ω_ϵ an ϵ-neighborhood of Ω. Then

$$\|u\|_{H^{m+2}(\Omega)} \leq C_\epsilon(\|u\|_{L^2(\Omega_\epsilon)} + \|f\|_{H^m(\Omega_\epsilon)}). \tag{1.1}$$