CHAPTER 1

Fourier transforms on the hyperbolic space

1. Basic geometry in the hyperbolic space

1.1. Upper-half space model. We begin with reviewing elementary geometric properties of the hyperbolic space $\mathbf{H}^n$. Throughout this note, $\mathbf{H}^n$ is the Euclidean upper-half space

$$\mathbf{R}^n_+ = \{(x,y)\ ;\ x \in \mathbf{R}^{n-1},\ y > 0\} \tag{1.1}$$

equipped with the metric

$$ds^2 = \frac{|dx|^2 + (dy)^2}{y^2}. \tag{1.2}$$

In the following, for $v = (v_1, \cdots, v_d) \in \mathbf{R}^d$, $|v|$ means its Euclidean length : $|v| = \left(\sum_{i=1}^d v_i^2\right)^{1/2}$.

Theorem 1.1. *(1) The following 4 maps are the generators of the group of isometries on* $\mathbf{H}^n$ *:*
(a) dilation : $(x,y) \to (\lambda x, \lambda y),\ \lambda > 0$,
(b) translation : $(x,y) \to (x+v, y),\ v \in \mathbf{R}^{n-1}$,
(c) rotation : $(x,y) \to (Rx, y),\ R \in O(n-1)$,
(d) inversion with respect to the unit sphere centered at $(0,0)$ *:*

$$(x,y) \to (\overline{x}, \overline{y}) = \frac{(x,y)}{|x|^2 + |y|^2}.$$

(2) Any isometry on $\mathbf{H}^n$ *is a product of the above 4 isometries.*

Proof. The assertion (1) follows from a direct computation. We use

$$d\overline{x} = \frac{dx}{r^2} - \frac{2x}{r^3}dr, \quad d\overline{y} = \frac{dy}{r^2} - \frac{2y}{r^3}dr,$$

where $r^2 = x^2 + y^2$, $\overline{x} = x/r^2$, $\overline{y} = y/r^2$, to prove (d). The proof of the assertion (2) is in [**15**] pp. 21, 24. □

Recall that the inversion with respect to the sphere $\{|x - x_0| = r\}$ is the map: $x \to r^2(x - x_0)/|x - x_0|^2 + x_0$. We give examples of the isometry in $\mathbf{H}^2$ and $\mathbf{H}^3$, which can be proved by a straightforward computation.

1.2. $\mathbf{H}^2$ and linear fractional transformation. When $n = 2$, it is convenient to identify a point $(x,y) \in \mathbf{H}^2$ with the complex number $z = x + iy$. For a matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}),$$