Introduction

0.1. Fourier analysis on manifolds. The Fourier transform on $L^2(\mathbf{R}^n)$ and its inversion formula are well-known :

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x)dx, \tag{0.1}$$

$$f(x) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \widehat{f}(\xi)d\xi. \tag{0.2}$$

Since $-\Delta e^{ix\cdot\xi} = |\xi|^2 e^{ix\cdot\xi}$, $e^{ix\cdot\xi}$ is an eigenfunction of $-\Delta$. Therefore (0.1) and (0.2) illustrate the expansion of arbitrary functions in terms of eigenfunctions (more appropriately *generalized eigenfunctions* since they do not belong to $L^2(\mathbf{R}^n)$) of the Laplacian.

There are two directions of development of the above fact. One is quantum mechanics, where the Schrödinger operator $H = -\Delta + V(x)$ is the most basic tool to decribe the physical system of atoms or molecules. If H has the continuous spectrum, it is known that there exists a system of generalized eigenfunctions of H which play the same role as $e^{ix\cdot\xi}$. Moreover, by using these generalized eigenfunctions one can define an operator called the scattering matrix or the S-matrix, which is the fundamental object to study the physical properties of quantum mechanical particles through the scattering experiment.

The other direction is the Fourier transform on manifolds, especially on homogeneous spaces of Lie groups, which is a central theme in the representation theory. Hyperbolic manifolds, one of the deepest sources of classical mathematics, appear also in this context. In particular, hyperbolic quotient manifolds by the action of discrete subgroups of $SL(2, \mathbf{R})$ and the associated S-matrix are important objects in number theory.

0.2. Perturbation of the continuous spectrum. The aim of the perturbation theory of continuous spectrum is, given an operator H_0 whose spectral property is rather easy to understand, to study the spectral properties of $H_0 + V$, where V is the perturbation deforming the operator H_0. When $H = H_0 + V$ has the continuous spectrum, an effective way of studying its spectral properties is to construct a generalized Fourier tranform associated with H. To accomplish this idea, it is necessary that the Fourier transform for H_0 can be constructed easily. For example, it is the case for the Laplacian $-\Delta$ on $\mathbf{R}^n$. If the perturbation term V is an operator on the same Hilbert space as for H_0 and is not so strong, one can construct the Fourier transform associated with $H_0 + V$ by using the technique of functional analysis and partial differential equations.

This is not so easy for operators on hyperbolic manifolds. Even the construction of the Fourier transform associated with the Laplace-Beltrami operator on the hyperbolic space is no longer a trivial work. To construct the Fourier transform on hyperbolic spaces based on the upper half space model or the ball model, one needs deep knowledge of Bessel functions. Under the action of discrete subgroups, the properties of groups will reflect on the structure of manifolds or the construction of generalized eigenfunctions.