Appendix

A.1 Several facts from probability theory

In this section, we gather several facts from probability theory that are necessary in this monograph.

A.1.1 Convergence of probability measures

Let (S, d) be a metric space and $\mathcal{B}(S)$ the Borel σ-algebra of S, i.e., the smallest σ-algebra on S containing all open sets of S. (In this monograph, $S = \mathbb{R}^d$ or $\mathbb{C}$ in most cases.) By a *probability measure* on S we mean a measure on $\big(S, \mathcal{B}(S)\big)$ with total measure 1. For simplicity, put

$$\begin{aligned}\mathcal{P}(S) &:= \text{the set of all probability measures on } S,\\ C_{\mathrm{b}}(S) &:= \text{the set of all bounded continuous functions of } S \text{ to } \mathbb{R}.\end{aligned}$$

Definition A.1 Let $\nu_n \in \mathcal{P}(S)$ $(n \geq 1)$ and $\nu \in \mathcal{P}(S)$. Then

$$\begin{aligned}&\nu_n \to \nu \text{ weakly as } n \to \infty\\ &\underset{\text{def}}{\Longleftrightarrow} \int_S f(x)\nu_n(dx) \to \int_S f(x)\nu(dx) \quad \text{as } n \to \infty \quad \text{for } ^{\forall} f \in C_{\mathrm{b}}(S).\end{aligned}$$

In this case, we say that ν_n *converges weakly* to ν as $n \to \infty$.

Claim A.1 *Let* $\nu_n \in \mathcal{P}(S)$ $(n \geq 1)$ *and* $\nu \in \mathcal{P}(S)$. *The following conditions* (i) $\sim$ (iv) *are equivalent to each other:*

(i) $\nu_n \to \nu$ *weakly as* $n \to \infty$,

(ii) *For every closed set* F *of* S, $\overline{\lim_{n\to\infty}}\, \nu_n(F) \leq \nu(F)$,

(iii) *For every open set* O *of* S, $\underline{\lim_{n\to\infty}}\, \nu_n(O) \geq \nu(O)$,

(iv) *For every continuity set* B *of* ν, *i.e.,* $B \in \mathcal{B}(S)$ *satisfying* $\nu(\partial B) = 0$, $\lim_{n\to\infty} \nu_n(B) = \nu(B)$.

For the proof, cf. Kotani [20, Proposition 9.2], H. Sato [29, Theorem 11.2], Stroock [31, Theorem 3.1.5].