Chapter 6

Some facts from analytic number theory

The aim of this chapter is to prove Claim 5.5 which was used in proving the main theorem, i.e., the Bohr-Jessen limit theorem. Following Matsumoto [26], we show a general theorem, i.e., Carlson's mean value theorem $\left[\text{cf. Theorem 6.3}\right]$, and then apply it to prove this claim.

For the proof of this claim and Carlson's mean value theorem, we study the following matters:

- Square mean value estimate of the Riemann zeta function, in other words, asymptotics of $\int_1^T |\zeta(\sigma+\sqrt{-1}t)|^2 dt$ as $T \to \infty$,
- Exponential decay of $\Gamma^{(l)}(\sigma+\sqrt{-1}t)$ as $|t| \to \infty$, where $\Gamma^{(l)}$ is the lth derivative of the gamma function.

These are discussed in Sections 6.1 and 6.2 respectively. After that, in Section 6.3, we present Carlson's mean value theorem and give its proof. Finally, in Section 6.4, we prove Claim 5.5, which is quickly finished owing to considerable efforts up to then.

6.1 Square mean value estimate of $\zeta(s)$

We begin with an easy part of the square mean value estimate.

Claim 6.1
$$\left|\int_1^T \left|\zeta(\sigma+\sqrt{-1}t)\right|^2 dt - T\zeta(2\sigma)\right|$$
$$\leq \zeta(2\sigma) + 8\zeta(2\sigma-1) - 8\zeta'(2\sigma-1) + 4\frac{\zeta(\sigma)^2}{\log 2}, \quad T \geq 1, \sigma > 1.$$

Proof. Fix $\sigma > 1$. For each $t \in \mathbb{R}$,

$$\begin{aligned}\left|\zeta(\sigma+\sqrt{-1}t)\right|^2 &= \zeta(\sigma+\sqrt{-1}t)\overline{\zeta(\sigma+\sqrt{-1}t)}\\ &= \sum_{m=1}^{\infty}\frac{1}{m^{\sigma+\sqrt{-1}t}}\sum_{n=1}^{\infty}\frac{1}{n^{\sigma-\sqrt{-1}t}}\\ &= \sum_{n,m=1}^{\infty}\frac{n^{\sqrt{-1}t}m^{-\sqrt{-1}t}}{n^{\sigma}m^{\sigma}}\end{aligned}$$