Chapter 5

Bohr-Jessen limit theorem

5.1 Log zeta function

To state the limit theorem, we begin with the definition of the log zeta function. We intend to make its definition given by Matsumoto [cf. [26, Chapter 6]] clear.

Claim 5.1 *Put* $\Gamma^{\dagger 1} := \{t \in \mathbb{R} \setminus \{0\};\, {}^{\exists}\sigma \in (0,1)\ \textit{s.t.}\ \zeta(\sigma + \sqrt{-1}t) = 0\}$. *Then*

$$^{\forall}t \in \mathbb{R},\ {}^{\exists}\delta > 0\ \textit{s.t.}\ \#\big(U_\delta(t) \cap \Gamma\big) \le 1,$$

where $U_\delta(t) = (t - \delta, t + \delta)$.

Proof. We prove it by a reduction to absurdity. Assume that

$$^{\exists}t \in \mathbb{R}\ \text{ s.t. }\ \#\big(U_\delta(t) \cap \Gamma\big) \ge 2 \quad ({}^{\forall}\delta > 0).$$

Then, for ${}^{\forall}n \in \mathbb{N}$, ${}^{\exists}t_n \in \Gamma$ s.t. $0 < |t_n - t| < \frac{1}{n}$. By the definition of Γ, ${}^{\exists}\sigma_n \in (0,1)$ s.t. $\zeta(\sigma_n + \sqrt{-1}t_n) = 0$. Since $\{\sigma_n\}_{n=1}^{\infty}$ is bounded, ${}^{\exists}\{n'\}$: a subsequence, ${}^{\exists}\sigma \in [0,1]$ s.t. $\sigma_{n'} \to \sigma$.

In case $\sigma \neq 1$ or $\sigma = 1$ and $t \neq 0$,

$$\begin{aligned}
&\sigma_{n'} + \sqrt{-1}t_{n'} \to \sigma + \sqrt{-1}t \in \mathbb{C} \setminus \{1\},\\
&\sigma_{n'} + \sqrt{-1}t_{n'} \neq \sigma + \sqrt{-1}t \quad ({}^{\forall}n') \quad \big[\because\ t_{n'} \neq t\big],\\
&\{\sigma_{n'} + \sqrt{-1}t_{n'}\}_{n'} \subset \mathbb{C} \setminus \{1\} \quad \big[\because\ \sigma_{n'} \in (0,1)\big],\\
&\zeta(\sigma_{n'} + \sqrt{-1}t_{n'}) = 0 \quad ({}^{\forall}n').
\end{aligned}$$

By the uniqueness theorem, this implies that $\zeta(\cdot) = 0$ on $\mathbb{C} \setminus \{1\}$.

In case $\sigma = 1$ and $t = 0$,

$$\begin{aligned}
&\sigma_{n'} + \sqrt{-1}t_{n'} \neq 1 \quad ({}^{\forall}n') \quad \big[\because\ \sigma_{n'} \in (0,1)\big],\\
&\zeta(\sigma_{n'} + \sqrt{-1}t_{n'}) = 0 \quad ({}^{\forall}n'),\\
&\sigma_{n'} + \sqrt{-1}t_{n'} \to 1.
\end{aligned}$$

[†1] Here Γ is a subset of $\mathbb{R}$. It is entirely a different thing from the gamma function.