Chapter 4

Riemann zeta function

In this chapter, following Kanemitsu [17], let us view the Riemann zeta function. For those readers who are not familiar with this, we here give detailed proofs for almost all. The matter in this chapter is necessary for the Bohr-Jessen limit theorem stated in the next chapter. Since this limit theorem is concerned with the Riemann zeta function (precisely the log zeta function), we may well view this function here.

4.1 Euler-Maclaurin summation formula

Definition 4.1 We define the *Bernoulli number* B_n $(n \geq 0)$ by

$$\begin{cases} B_0 := 1, \\ B_n := \dfrac{-1}{n+1} \displaystyle\sum_{0 \leq k < n} \binom{n+1}{k} B_k \quad (n \geq 1). \end{cases}$$

We call B_n the nth Bernoulli number.

Claim 4.1 (i) $\displaystyle\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0$ $(n \geq 2)$.

(ii) $\dfrac{z}{e^z - 1} = \displaystyle\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$ $(z \in \mathbb{C}$ *with* $|z| < 2\pi)$.

(iii) $B_1 = -\dfrac{1}{2}$, $B_2 = \dfrac{1}{6}$, $B_n = 0$ $({}^\forall n \in 2\mathbb{N} + 1)$.

Proof. (i) For $n \geq 2$,

$$\begin{aligned} \sum_{k=0}^{n-1} \binom{n}{k} B_k &= \sum_{0 \leq k < n-1} \binom{n}{k} B_k + \binom{n}{n-1} B_{n-1} \\ &= \sum_{0 \leq k < n-1} \binom{n}{k} B_k + n \cdot \frac{-1}{n} \sum_{0 \leq k < n-1} \binom{n}{k} B_k \\ &= 0. \end{aligned}$$