Chapter 3

Complex random variable $\sum_p -\log(1-\frac{e(-\log p)}{p^\sigma})$ on $(\mathbb{R}^{\mathbb{B}}, \mathbf{P})$

In this chapter, we define the complex random variable in the heading above, whose distribution is the limit distribution in the Bohr-Jessen limit theorem.

3.1 Complex random variables $e(\lambda)$

Definition 3.1 For $\lambda \in \mathbb{R} \setminus \{0\}$, we define

$$\begin{array}{cccc} e(\lambda): & \mathbb{R}^{\mathbb{B}} & \to & \mathbb{C} \\ & \Cup & & \Cup \\ & (x_f)_{f\in\mathbb{B}} & \mapsto & x_{\cos\lambda\cdot} + \sqrt{-1}x_{\sin\lambda\cdot}, \end{array}$$

where $\cos\lambda\cdot$ and $\sin\lambda\cdot$ denote almost periodic functions $t \mapsto \cos\lambda t$ and $t \mapsto \sin\lambda t$, respectively.

Note that $e(\lambda) = \pi_{\cos\lambda\cdot} + \sqrt{-1}\pi_{\sin\lambda\cdot}$.

Claim 3.1 *For $f : \mathbb{C} \to \mathbb{C}$ bounded Borel measurable,*

$$E^{\mathbf{P}}\big[f\big(e(\lambda)\big)\big] = \frac{1}{2\pi}\int_0^{2\pi} f\big(e^{\sqrt{-1}t}\big)dt.$$

Proof. We divide the proof into three steps:
<u>1°</u> For $f \in C_b(\mathbb{C};\mathbb{C})$, i.e., bounded continuous function $f : \mathbb{C} \to \mathbb{C}$,

$$\begin{aligned} E^{\mathbf{P}}\big[f\big(e(\lambda)\big)\big] &= E^{\mathbf{P}}\Big[f\big(\pi_{\cos\lambda\cdot} + \sqrt{-1}\pi_{\sin\lambda\cdot}\big)\Big] \\ &= \iint_{\mathbb{R}^2} f(a+\sqrt{-1}b)\mathbf{P}\circ\pi^{-1}_{(\cos\lambda\cdot,\sin\lambda\cdot)}(dadb) \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\cos\lambda t + \sqrt{-1}\sin\lambda t)dt \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f\big(e^{\sqrt{-1}\lambda t}\big)dt \end{aligned}$$