Chapter 2

Probability measure P on $\mathbb{R}^{\mathbb{B}}$

Topics of this chapter come from Fukuyama [11]. To tell the truth, Chapter 1 was prepared, since we wanted the fact that every almost periodic function always has the mean value. Based on these mean values, we define a probability measure **P** on the space $\mathbb{R}^{\mathbb{B}}$ of large volume.

2.1 Definition of the probability measure P

Definition 2.1 $\mathbb{B} := AP(\mathbb{R}) \cap C(\mathbb{R}; \mathbb{R})$, i.e., $\mathbb{B}$ is the set of all real-valued almost periodic functions.

Definition 2.2 For $T > 0$, we define a probability measure P_T on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

$$P_T(E) := \frac{1}{2T}\mu\big([-T, T] \cap E\big), \quad E \in \mathcal{B}(\mathbb{R}).$$

Here μ is the 1-dimensional Lebesgue measure.

Lemma 2.1 *For $f_1, \dots, f_n \in \mathbb{B}$, let $P_T^{(f_1,\dots,f_n)}$ be an image measure of P_T by the continuous mapping*

$$\mathbb{R} \ni t \mapsto (f_1, \dots, f_n)(t) := \big(f_1(t), \dots, f_n(t)\big) \in \mathbb{R}^n.$$

($P_T^{(f_1,\dots,f_n)}$ is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.) Then

$$\begin{aligned} &{}^{\exists} P\text{: a probability measure on } \big(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\big) \\ &\text{s.t. } P_T^{(f_1,\dots,f_n)} \to P \text{ weakly as } T \to \infty. \end{aligned}$$

Proof. Let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. By Claim 1.3 and Claim 1.1(iii), $e^{\sqrt{-1}\sum_{i=1}^n \xi_i f_i(\cdot)} \in AP(\mathbb{R})$. Thus, by Theorem 1.1,

$$\begin{aligned} \widehat{P_T^{(f_1,\dots,f_n)}}(\xi)^{\dagger 1} &= \int_{\mathbb{R}^n} e^{\sqrt{-1}\langle \xi, x\rangle} P_T^{(f_1,\dots,f_n)}(dx) \\ &= \int_{\mathbb{R}} e^{\sqrt{-1}\sum_{i=1}^n \xi_i f_i(t)} P_T(dt) \end{aligned}$$