Introduction

We would describe more explicitly what was stated in Preface.

The Riemann zeta function $\zeta(\cdot)$ is defined on the half-plane $\operatorname{Re} s > 1$ by the Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ where $n^s = e^{s \log n}$. This function is holomorphic there and is also expressed by the Euler product $\prod_{p:\text{prime}} \frac{1}{1-\frac{1}{p^s}}$. Moreover $\zeta(\cdot)$ is analytically continuable to a meromorphic function on the whole complex plane $\mathbb{C}$ that is holomorphic except $s = 1$ and has a simple pole at $s = 1$ with residue 1. Let us denote this meromorphic function by the same $\zeta(\cdot)$.

By the Euler product above, $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$. It is shown that $\zeta(s) \neq 0$ on the line $\operatorname{Re} s = 1$, and thus there are no zeros of $\zeta(\cdot)$ in the closed half-plane $\operatorname{Re} s \geq 1$. From the functional equation of $\zeta(\cdot)$, it is seen that its zeros in the closed half-plane $\operatorname{Re} s \leq 0$ are negative even integers $-2, -4, -6, \ldots$, which are called the *trivial zeros*. The Riemann hypothesis is concerned with *non-trivial zeros*, namely the zeros in the strip $0 < \operatorname{Re} s < 1$, and states that all non-trivial zeros are on the line $\operatorname{Re} s = \frac{1}{2}$. Since $\zeta(s) \neq 0$ for $\operatorname{Re} s > \frac{1}{2}$ under this hypothesis, the log zeta function $\log \zeta$ can be defined as a holomorphic function on the domain G', the half-plane $\operatorname{Re} s > \frac{1}{2}$ excluding the line segment $(\frac{1}{2}, 1]$ with the derivative $\frac{\zeta'(s)}{\zeta(s)}$ and the value $\log \frac{\pi^2}{6}$ at $s = 2$. In other words, $\log \zeta$ is a primitive function of $\frac{\zeta'}{\zeta}$ on G' with the value $\log \frac{\pi^2}{6}$ at $s = 2$. Unfortunately, the Riemann hypothesis is not yet proved to hold valid at the moment, so that a modification of this definition is necessary. In fact, in place of G', we have only to take a simply connected domain G of $\mathbb{C}$ such that G contains the half-plane $\operatorname{Re} s > 1$ but no zeros of $\zeta(\cdot)$.

When $\sigma > 1$, it is easily seen from the Dirichlet series expression of $\zeta(\cdot)$ that $|\zeta(\sigma + \sqrt{-1}t)| \leq \zeta(\sigma)$ for all $t \in \mathbb{R}$. Thus $\zeta(\sigma + \sqrt{-1}t)$ takes the value in the closed disc with center at origin and radius $\zeta(\sigma)$. Then how is the case when $\sigma \leq 1$? Since, by the functional equation of $\zeta(\cdot)$, the value of $\zeta(1-s)$ is computed from that of $\zeta(s)$ immediately, we may restrict ourselves to the case when $\frac{1}{2} \leq \sigma \leq 1$. In the early 1910s, H. Bohr obtained many results about the behavior of $\zeta(\cdot)$ on the strip $\frac{1}{2} < \operatorname{Re} s \leq 1$. Among them, the following is an answer of the question above [cf. Bohr [2]]:

Theorem 1 *For $\frac{1}{2} < \sigma \leq 1$, the set $\{\log \zeta(\sigma + \sqrt{-1}t); t \in \mathbb{R}$ with $\sigma + \sqrt{-1}t \in G\}$ is dense in $\mathbb{C}$.*

In fact, since $\zeta(\sigma + \sqrt{-1}t) = e^{\log \zeta(\sigma + \sqrt{-1}t)}$, it is seen from the theorem that the set $\{\zeta(\sigma + \sqrt{-1}t); t \in \mathbb{R}\}$ is dense in $\mathbb{C}$ [cf. Bohr-Courant [4]]. According as $\sigma > 1$ or $\sigma \leq 1$, the behavior of $\zeta(\sigma + \sqrt{-1}\cdot)$, i.e., $\zeta(\cdot)$ on the line $\operatorname{Re} s = \sigma$ changes drastically.

Bohr much studied about the value-distribution of $\zeta(\cdot)$, which motivated him to develop the theory of almost periodic functions. It was in the 1930s that he, together with