Chapter 9

Not well-posed results

9.1 Introduction

For the second order differential operator in $\mathbb{R}^2$ with real analytic coefficient $a(x_0, x_1) \geq 0$ defined near the origin

$$P = -D_0^2 + a(x_0, x_1)D_1^2$$

the Cauchy problem is C^∞ well posed near the origin ([40]). Since then it has been conjectured that the Cauchy problem is C^∞ well posed for any second order differential operator of divergence form with real analytic coefficients

$$Pu = -D_0^2 u + \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x)D_{x_j}u), \quad a_{ij}(x) = a_{ji}(x)$$

where $a_{ij}(x)$ are real analytic and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0, \quad \forall \xi' = (\xi_1, ..., \xi_n) \in \mathbb{R}^n.$$

In Section 8.1 we have shown that the operator P_{mod} is of divergence form and hence this gives a counter example of the conjecture. In this chapter we show somewhat stronger assertion on the well-posedness of the Cauchy problem for P_{mod}, that is the Cauchy problem for $P_{mod} + Q$ is not $\gamma^{(s)}$ well posed for any $s > 6$ whatever the lower order term Q is. Recall that the coefficients of P_{mod} are not only real analytic but also polynomials. This is a quite unexpected fact. On the other hand note that the Cauchy problem for $P_{mod}+Q$ is $\gamma^{(s)}$ well posed for any $1 \leq s \leq 2$ and for any lower order term Q, which is a particular case of the general result proved in [9].

Let us consider again

$$P_{mod}(x, D) = -D_0^2 + 2x_1 D_0 D_2 + D_1^2 + x_1^3 D_2^2 \tag{9.1.1}$$

in $\mathbb{R}^3$. Then we have