Chapter 8

Optimality of the Gevrey index

8.1 Non solvability in C^∞ and the Gevrey class

In this chapter we study the following model operator

$$P_{mod}(x,D) = -D_0^2 + 2x_1 D_0 D_n + D_1^2 + x_1^3 D_n^2. \tag{8.1.1}$$

It is worthwhile to note that if we make the change of coordinates

$$y_j = x_j \ (0 \le j \le n-1), \ \ y_n = x_n + x_0 x_1$$

which preserves the initial plane $x_0 = const.$, the operator P_{mod} is written in these coordinates as

$$P_{mod} = -D_0^2 + (D_1 + x_0 D_n)^2 + (x_1\sqrt{1+x_1}D_n)^2 = -D_0^2 + A^2 + B^2.$$

Here we have $A^* = A$ and $B^* = B$ while $[D_0, A] \neq 0$ and $[A, B] \neq 0$.

Let us denote by $p(x,\xi)$ the symbol of $P_{mod}(x,D)$ then it is clear that the double characteristic manifold near the double characteristic point $\bar\rho = (0,(0,...,0,1)) \in \mathbb{R}^{2(n+1)}$ is given by

$$\Sigma = \{(x,\xi) \in \mathbb{R}^{2(n+1)} \mid \xi_0 = 0, x_1 = 0, \xi_1 = 0\}$$

and the localization of p at $\rho \in \Sigma$ is given by $p_\rho(x,\xi) = -\xi_0^2 + 2x_1\xi_0 + \xi_1^2$. This is just (2) in Theorem 2.3.1 with $k = l = 1$ where ξ_1 and x_1 is exchanged. Since $(x_1,\xi_1) \mapsto (\xi_1, -x_1)$ is a symplectic change of the coordinates system then we see

$$\mathrm{Ker}\, F_p^2(\rho) \cap \mathrm{Im}\, F_p^2(\rho) \neq \{0\}, \ \ \rho \in \Sigma.$$

The main feature of p is that the Hamilton flow H_p lands tangentially on Σ. Indeed the integral curve of H_p

$$\xi_1 = -\frac{x_0^2}{4}, \ x_n = \frac{x_0^5}{8}, \ \xi_0 = 0, \ \xi_1 = \frac{x_0^3}{8}, \ x_j, \xi_j = \text{constants}, \ |x_0| > 0$$