Chapter 6

Well-posedness in the Gevrey classes

6.1 Gevrey well-posedness

We study the same operator

$$P(x,D) = -D_0^2 + \sum_{|\alpha|\le 2, \alpha_0<2} a_\alpha(x) D^\alpha = P_2 + P_1 + P_0$$
$$= -D_0^2 + A_1(x,D')D_0 + A_2(x,D')$$

as in the preceding chapter. As before we assume that p vanishes exactly of order 2 on a C^∞ manifold Σ on which σ has constant rank and p is noneffectively hyperbolic, that is we assume that $\Sigma = \{(x,\xi) \mid p(x,\xi)=0, dp(x,\xi)=0\}$ is a C^∞ manifold and (4.1.1) is satisfied.

We assume (5.1.1) but not (5.1.2). Thus the Hamilton flow H_p may touch Σ tangentially. If the Hamilton map really touches Σ tangentially then the Cauchy problem is no more C^∞ well posed even though under the Levi condition (which will be proved in Chapter 8). What is the best we can expect is the well-posedness in much smaller function space, that is in the Gevrey class s with $1 \le s \le 5$ under the Levi condition. We start with the definition of the Gevrey classes.

Definition 6.1.1 *We say $f(x) \in \gamma^{(s)}(\mathbb{R}^n)$, the Gevrey class of order s (≥ 1) if for any compact set $K \subset \mathbb{R}^n$ there exist $C>0$, $h>0$ such that*

$$|\partial_x^\alpha f(x)| \le C h^{-|\alpha|} |\alpha|!^s, \quad x \in K, \ \forall \alpha \in \mathbb{N}^n$$

holds. We also set

$$\gamma_0^{(s)}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n) \cap \gamma^{(s)}(\mathbb{R}^n).$$

Theorem 6.1.1 *Assume (4.1.1), (5.1.1) and that $P_{sub} = 0$ everywhere on Σ. Then the Cauchy problem for P is well posed in $\gamma^{(s)}$ with $1 \le s \le 5$, that is*