Chapter 4

Noneffectively hyperbolic Cauchy problem I

4.1 C^∞ well-posedness

Let

$$P(x,D) = D_0^2 + \sum_{|\alpha|\le 2, \alpha_0<2} a_\alpha(x) D^\alpha = P_2 + P_1 + P_0$$

be a second order differential operator, defined in an open neighborhood of the origin of $\mathbb{R}^{n+1}$, hyperbolic with respect to the x_0 direction and with principal symbol $p(x,\xi)$ where $x = (x_0, x_1, ..., x_n)$, $\xi = (\xi_0, \xi_1, ..., \xi_n)$.

We now state more precisely our assumptions. We shall assume in the following that p vanishes exactly of order 2 on a C^∞ submanifold Σ on which σ has constant rank and p is noneffectively hyperbolic, that is we assume that $\Sigma = \{(x,\xi) \mid p(x,\xi) = 0, dp(x,\xi) = 0\}$ is a C^∞ manifold and

$$\text{(4.1.1)} \qquad \begin{cases} \mathrm{Sp}(F_p(\rho)) \subset i\mathbb{R}, \quad \rho \in \Sigma, \\ \dim T_\rho\Sigma = \dim \mathrm{Ker}\, F_p(\rho), \quad \rho \in \Sigma, \\ \mathrm{rank}\, \big(\sigma|_\Sigma\big) = \text{constant}, \quad \text{on } \Sigma \end{cases}$$

where $\mathrm{Sp}(F_p(\rho))$ denotes the spectrum of $F_p(\rho)$. According to the spectral structure of $F_p(\rho)$, two different possible cases may arise

$$\text{(4.1.2)} \qquad \mathrm{Ker}\, F_p^2(\rho) \cap \mathrm{Im}\, F_p^2(\rho) = \{0\}$$

and

$$\mathrm{Ker}\, F_p^2(\rho) \cap \mathrm{Im}\, F_p^2(\rho) \neq \{0\}$$

about which we made detailed studies in the previous chapter.

As shown in Proposition 3.2.1 if p verifies (4.1.2) then p admits an elementary decomposition. In this chapter, assuming (4.1.2), we prove that the Cauchy problem is C^∞ well posed deriving energy estimates via elementary decomposition under the Levi or the strict Ivrii-Petkov-Hörmander condition.