Chapter 1

Necessary conditions for well-posedness

1.1 Lax-Mizohata theorem

Let $P(x, D)$ be a differential operator of order m defined in a neighborhood Ω of the origin of $\mathbb{R}^{n+1}$ with coordinates $x = (x_0, x_1, ..., x_n) = (x_0, x')$

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \tag{1.1.1}$$

where $D^\alpha = D_0^{\alpha_0} \cdots D_n^{\alpha_n}$, $D_j = -i\partial/\partial x_j$ and $a_\alpha(x) \in C^\infty(\Omega)$. We assume that hyperplanes $x_0 =$ const. are non characteristic for P. Thus we may assume $a_{(m,0,...,0)}(x) = 1$.

Definition 1.1.1 *We say that the Cauchy problem for P is C^∞ well posed near the origin if there are $\epsilon > 0$ and a neighborhood ω of the origin such that for any $|\tau| \leq \epsilon$ and for any $f(x) \in C_0^\infty(\omega)$ vanishing in $x_0 < \tau$ there is a unique solution $u(x) \in H^\infty(\omega)$ to $Pu = f$ in ω vanishing in $x_0 < \tau$, where $H^\infty(\omega) = \cap_{p=0}^\infty H^p(\omega)$ and $H^p(\omega)$ denotes the usual Sobolev space of order p.*

Assume that $u \in H^\infty(\omega)$ vanishes in $x_0 < \tau$ with $|\tau| < \epsilon$. If $Pu = 0$ in $x_0 < t$ ($|t| < \epsilon$) then we conclude that $u = 0$ in $x_0 < t$. To see this, take $\chi \in C_0^\infty(\omega)$ and note that the equation $Pw = P(\chi u)$ has a solution $w \in H^\infty(\omega)$ vanishing in $x_0 < t$. Since $w - \chi u = 0$ in $x_0 < \min\{\tau, t\}$, and $P(w - \chi u) = 0$, by the uniqueness we get $w = \chi u$ and hence $u = 0$ in $x_0 < t$. Since $\chi \in C_0^\infty(\omega)$ is arbitrary we conclude $u = 0$ in $x_0 < t$.

Lemma 1.1.1 *Assume that the Cauchy problem for P is C^∞ well posed near the origin. Then the following classical Cauchy problem has a unique solution $u \in H^\infty(\omega)$*

$$\begin{cases} Pu = f \quad \textit{in} \quad \omega \cap \{x_0 > \tau\} \\ D_0^j u(\tau, x') = u_j(x'), \quad j = 0, 1, ..., m-1 \end{cases} \tag{1.1.2}$$