CHAPTER 12

Connection problem

12.1. Connection formula

For a realizable tuple $\mathbf{m} \in \mathcal{P}_{p+1}$, let $P_{\mathbf{m}}u = 0$ be a universal Fuchsian differential equation with the Riemann scheme

$$
\left\{\begin{matrix} x = 0 & c_1 = 1 & \cdots & c_j & \cdots & c_p = \infty \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{j,1}]_{(m_{j,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{j,n_j}]_{(m_{j,n_j})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{matrix}\right\}. \tag{12.1}
$$

The singular points of the equation are c_j for $j = 0, \ldots, p$. In this section we always assume $c_0 = 0$, $c_1 = 1$ and $c_p = \infty$ and $c_j \notin [0, 1]$ for $j = 2, \ldots, p-1$. We also assume that $\lambda_{j,\nu}$ are generic.

Definition 12.1 (connection coefficients). Suppose $\lambda_{j,\nu}$ are generic under the Fuchs relation. Let $u_0^{\lambda_{0,\nu_0}}$ and $u_1^{\lambda_{1,\nu_1}}$ be normalized local solutions of $P_{\mathbf{m}} = 0$ at $x = 0$ and $x = 1$ corresponding to the exponents λ_{0,ν_0} and λ_{1,ν_1}, respectively, so that $u_0^{\lambda_{0,\nu_0}} \equiv x^{\lambda_{0,\nu_0}} \mod x^{\lambda_{0,\nu_0}+1}\mathcal{O}_0$ and $u_1^{\lambda_{1,\nu_1}} \equiv (1-x)^{\lambda_{1,\nu_1}} \mod (1-x)^{\lambda_{1,\nu_1}+1}\mathcal{O}_1$. Here $1 \le \nu_0 \le n_0$ and $1 \le \nu_1 \le n_1$. If $m_{0,\nu_0} = 1$, $u_0^{\lambda_{0,\nu_0}}$ is uniquely determined and then the analytic continuation of $u_0^{\lambda_{0,\nu_0}}$ to $x = 1$ along $(0,1) \subset \mathbb{R}$ defines a *connection coefficient* with respect to $u_1^{\lambda_{1,\nu_1}}$, which is denoted by $c(0\!:\!\lambda_{0,\nu_0} \rightsquigarrow 1\!:\!\lambda_{1,\nu_1})$ or simply by $c(\lambda_{0,\nu_0} \rightsquigarrow \lambda_{1,\nu_1})$. The connection coefficient $c(1\!:\!\lambda_{1,\nu_1} \rightsquigarrow 0\!:\!\lambda_{0.\nu_0})$ or $c(\lambda_{1,\nu_1} \rightsquigarrow \lambda_{0.\nu_0})$ of $u_1^{\lambda_{1,\nu_1}}$ with respect to $u_0^{\lambda_{0,\nu_0}}$ are similarly defined if $m_{1,\nu_1} = 1$.

Moreover we define $c(c_i : \lambda_{i,\nu_i} \rightsquigarrow c_j : \lambda_{j,\nu_j})$ by using a suitable linear fractional transformation T of $\mathbb{C} \cup \{\infty\}$ which transforms $\{c_i, c_j\}$ to $\{0, 1\}$ so that $T(c_\nu) \notin (0, 1)$ for $\nu = 0, \ldots, p$. If $p = 2$, we define the map T so that $T(c_k) = \infty$ for the other singular point c_k. For example if $c_j \notin [0, 1]$ for $j = 2, \ldots, p-1$, we put $T(x) = \frac{x}{x-1}$ to define $c(0 : \lambda_{0,\nu_0} \rightsquigarrow \infty : \lambda_{p,\nu_p})$ or $c(\infty : \lambda_{p,\nu_p} \rightsquigarrow 0 : \lambda_{0,\nu_0})$.

In the definition $u_0^{\lambda_{0,\nu_0}}(x) = x^{\lambda_{0,\nu_0}}\phi(x)$ with analytic function $\phi(x)$ at 0 which satisfies $\phi(0) = 1$ and if $\operatorname{Re}\lambda_{1,\nu_1} < \operatorname{Re}\lambda_{1,\nu}$ for $\nu \ne \nu_1$, we have

$$
c(\lambda_{0,\nu_0} \rightsquigarrow \lambda_{1,\nu_1}) = \lim_{x \to 1-0} (1-x)^{-\lambda_{1,\nu_1}} u_0^{\lambda_{0,\nu_0}}(x) \qquad (x \in [0, 1)) \tag{12.2}
$$

by the analytic continuation. The connection coefficient $c(\lambda_{0.\nu_0} \rightsquigarrow \lambda_{1,\nu_1})$ meromorphically depends on spectral parameters $\lambda_{j,\nu}$. It also holomorphically depends on accessory parameters g_i and singular points $\frac{1}{c_j}$ $(j = 2, \ldots, p-1)$ in a neighborhood of given values of parameters.

The main purpose in this section is to get the explicit expression of the connection coefficients in terms of gamma functions when $\mathbf{m}$ is rigid and $m_{0,\nu} = m_{1,\nu'} = 1$.

Fist we prove the following key lemma which describes the effect of a middle convolution on connection coefficients.