CHAPTER 10

Reducibility

We examine the condition for the decomposition $P_{\mathbf{m}} = P_{\mathbf{m}'}P_{\mathbf{m}''}$ of universal operators with or without fixing the characteristic exponents (cf. Theorem 4.19 i)), which implies the reducibility of the equation $P_{\mathbf{m}}u = 0$. Note that the irreducibility of a Fuchsian differential equation equals the irreducibility of the monodromy of the equation and that it is kept under our reduction of the equation. In §10.2 we study the value of spectral parameters which makes the equation reducible and obtain Theorem 10.10. In particular we have a necessary and sufficient condition on characteristic exponents so that the monodromy of the solutions of the equation $P_{\mathbf{m}}u = 0$ with a rigid spectral type $\mathbf{m}$ is irreducible, which is given in Theorem 10.13.

10.1. Direct decompositions

For a realizable $(p+1)$-tuple $\mathbf{m} \in \mathcal{P}_{p+1}^{(n)}$, Theorem 6.14 gives the universal Fuchsian differential operator $P_{\mathbf{m}}(\lambda_{j,\nu}, g_i)$ with the Riemann scheme (4.15). Here $g_1, \dots, g_N$ are accessory parameters and $N = \operatorname{Ridx}\mathbf{m}$.

First suppose $\mathbf{m}$ is basic. Choose positive numbers n', n'', $m'_{j,1}$ and $m''_{j,1}$ such that

$$\begin{gathered} n = n' + n'', \quad 0 < m'_{j,1} \le n', \quad 0 < m''_{j,1} \le n'', \\ m'_{0,1} + \cdots + m'_{p,1} \le (p-1)n', \quad m''_{0,1} + \cdots + m''_{p,1} \le (p-1)n''. \end{gathered} \tag{10.1}$$

We choose other positive integers $m'_{j,\nu}$ and $m''_{j,\nu}$ so that $\mathbf{m}' = \bigl(m'_{j,\nu}\bigr)$ and $\mathbf{m}'' = \bigl(m''_{j,\nu}\bigr)$ are monotone tuples of partitions of n' and n'', respectively, and moreover

$$\mathbf{m} = \mathbf{m}' + \mathbf{m}''. \tag{10.2}$$

Theorem 6.6 shows that $\mathbf{m}'$ and $\mathbf{m}''$ are realizable. If $\{\lambda_{j,\nu}\}$ satisfies the Fuchs relation

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m'_{j,\nu}\lambda_{j,\nu} = n' - \frac{\operatorname{idx}\mathbf{m}'}{2} \tag{10.3}$$

for the Riemann scheme $\bigl\{[\lambda_{j,\nu}]_{(m'_{j,\nu})}\bigr\}$, Theorem 4.19 shows that the operators

$$P_{\mathbf{m}''}(\lambda_{j,\nu} + m'_{j,\nu} - \delta_{j,0}(p-1)n', g''_i) \cdot P_{\mathbf{m}'}(\lambda_{j,\nu}, g'_i) \tag{10.4}$$

has the Riemann scheme $\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\}$. Hence the equation $P_{\mathbf{m}}(\lambda_{j,\nu}, g_i)u = 0$ is not irreducible when the parameters take the values corresponding to (10.4).

In this section, we study the condition

$$\operatorname{Ridx}\mathbf{m} = \operatorname{Ridx}\mathbf{m}' + \operatorname{Ridx}\mathbf{m}'' \tag{10.5}$$

for realizable tuples $\mathbf{m}'$ and $\mathbf{m}''$ with $\mathbf{m} = \mathbf{m}' + \mathbf{m}''$. Under this condition the Fuchs relation (10.3) assures that the universal operator is reducible for any values of accessory parameters.