CHAPTER 8

Expression of local solutions

Fix $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p\\ 1\le\nu\le n_j}} \in \mathcal{P}_{p+1}$. Suppose $\mathbf{m}$ is monotone and irreducibly realizable. Let $P_{\mathbf{m}}$ be the universal operator with the Riemann scheme (4.15), which is given in Theorem 6.14. Suppose $c_1 = 0$ and $m_{1,n_1} = 1$. We give expressions of the local solution of $P_{\mathbf{m}}u = 0$ at $x = 0$ corresponding to the characteristic exponent λ_{1,n_1}.

THEOREM 8.1. *Retain the notation above and in* Definition 5.12. *Suppose* $\lambda_{j,\nu}$ *are generic. Let*

$$v(x) = \sum_{\nu=0}^{\infty} C_\nu x^{\lambda(K)_{1,n_1}+\nu} \tag{8.1}$$

be the local solution of $\big(\partial_{\max}^{K} P_{\mathbf{m}}\big)v = 0$ *at* $x = 0$ *with the condition* $C_0 = 1$. *Put*

$$\lambda(k)_{j,max} = \lambda(k)_{j,\ell(k)_j}. \tag{8.2}$$

Note that if $\mathbf{m}$ *is rigid, then*

$$v(x) = x^{\lambda(K)_{1,n_1}} \prod_{j=2}^{p} \Big(1 - \frac{x}{c_j}\Big)^{\lambda(K)_{j,max}}. \tag{8.3}$$

The function

$$\begin{aligned} u(x) &:= \prod_{k=0}^{K-1} \frac{\Gamma\big(\lambda(k)_{1,n_1} - \lambda(k)_{1,max} + 1\big)}{\Gamma\big(\lambda(k)_{1,n_1} - \lambda(k)_{1,max} + \mu(k) + 1\big)\Gamma\big(-\mu(k)\big)} \\ &\quad \int_0^{s_0} \cdots \int_0^{s_{K-1}} \prod_{k=0}^{K-1} (s_k - s_{k+1})^{-\mu(k)-1} \\ &\quad \cdot \prod_{k=0}^{K-1} \Big(\Big(\frac{s_k}{s_{k+1}}\Big)^{\lambda(k)_{1,max}} \prod_{j=2}^{p} \Big(\frac{1 - c_j^{-1}s_k}{1 - c_j^{-1}s_{k+1}}\Big)^{\lambda(k)_{j,max}} \Big) \\ &\quad \cdot v(s_K)ds_K \cdots ds_1 \Big|_{s_0 = x} \end{aligned} \tag{8.4}$$

is the solution of $P_{\mathbf{m}}u = 0$ *so normalized that* $u(x) \equiv x^{\lambda_{1,n_1}} \mod x^{\lambda_{1,n_1}+1}\mathcal{O}_0$.

Here we note that

$$\begin{aligned} &\prod_{k=0}^{K-1} \Big(\Big(\frac{s_k}{s_{k+1}}\Big)^{\lambda(k)_{1,max}} \prod_{j=2}^{p} \Big(\frac{1 - c_j^{-1}s_k}{1 - c_j^{-1}s_{k+1}}\Big)^{\lambda(k)_{j,max}} \Big) \\ &= \frac{s_0^{\lambda(0)_{1,max}}}{s_K^{\lambda(K-1)_{1,max}}} \prod_{j=1}^{p} \frac{(1 - c_j^{-1}s_0)^{\lambda(0)_{j,max}}}{(1 - c_j^{-1}s_K)^{\lambda(K-1)_{j,max}}} \\ &\quad \cdot \prod_{k=1}^{K-1} \Big(s_k^{\lambda(k)_{1,max} - \lambda(k-1)_{1,max}} \prod_{j=2}^{p} (1 - c_j^{-1}s_k)^{\lambda(k)_{j,max} - \lambda(k-1)_{j,max}} \Big). \end{aligned} \tag{8.5}$$