CHAPTER 6

Deligne-Simpson problem

In this chapter we give an answer for the existence and the construction of Fuchsian differential equations with given Riemann schemes and examine the irreducibility for generic spectral parameters.

6.1. Fundamental lemmas

First we prepare two lemmas to construct Fuchsian differential operators with a given spectral type.

DEFINITION 6.1. For $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p\\ 1\le\nu\le n_j}} \in \mathcal{P}_{p+1}^{(n)}$, we put

$$\begin{aligned}N_\nu(\mathbf{m}) := {}&(p-1)(\nu+1)+1\\ &-\#\{(j,i)\in\mathbb{Z}^2\,;\,i\ge 0,\ 0\le j\le p,\ \widetilde{m}_{j,i}\ge n-\nu\},\end{aligned} \tag{6.1}$$

$$\widetilde{m}_{j,i} := \sum_{\nu=1}^{n_j}\max\big\{m_{j,\nu}-i,0\big\}. \tag{6.2}$$

See the Young diagram in (6.32) and its explanation for an interpretation of the number $\widetilde{m}_{j,i}$.

LEMMA 6.2. *We assume that* $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p\\ 1\le\nu\le n_j}} \in \mathcal{P}_{p+1}^{(n)}$ *satisfies*

$$m_{j,1}\ge m_{j,2}\ge\cdots\ge m_{j,n_j}>0 \quad\textit{and}\quad n>m_{0,1}\ge m_{1,1}\ge\cdots\ge m_{p,1} \tag{6.3}$$

and

$$m_{0,1}+\cdots+m_{p,1}\le(p-1)n. \tag{6.4}$$

Then

$$N_\nu(\mathbf{m})\ge 0 \qquad (\nu=2,3,\dots,n-1) \tag{6.5}$$

if and only if $\mathbf{m}$ *is not any one of*

$$\begin{gathered}(k,k;k,k;k,k;k,k),\quad (k,k,k;k,k,k;k,k,k),\\ (2k,2k;k,k,k,k;k,k,k,k)\\ \textit{and}\ \ (3k,3k;2k,2k,2k;k,k,k,k,k,k)\ \ \textit{with}\ \ k\ge 2.\end{gathered} \tag{6.6}$$

PROOF. Put

$$\begin{aligned}\phi_j(t) &:= \sum_{\nu=1}^{n_j}\max\{m_{j,\nu}-t,0\},\\ \bar\phi_j(t) &:= n\Big(1-\frac{t}{m_{j,1}}\Big)\quad\text{for}\quad j=0,\dots,p.\end{aligned}$$