CHAPTER 5

Reduction of Fuchsian differential equations

Additions and middle convolutions introduced in Chapter 1 are transformations within Fuchsian differential operators and in this chapter we examine how their Riemann schemes change under the transformations.

PROPOSITION 5.1. i) *Let $Pu = 0$ be a Fuchsian differential equation. Suppose there exists $c \in \mathbb{C}$ such that $P \in (\partial - c)W[x]$. Then $c = 0$.*

ii) *For $\phi(x) \in \mathbb{C}(x)$, $\lambda \in \mathbb{C}$, $\mu \in \mathbb{C}$ and $P \in W[x]$, we have*

$$P \in \mathbb{C}[x]\,\mathrm{RAdei}\bigl(-\phi(x)\bigr) \circ \mathrm{RAdei}\bigl(\phi(x)\bigr)P, \tag{5.1}$$

$$P \in \mathbb{C}[\partial]\,\mathrm{RAd}\bigl(\partial^{-\mu}\bigr) \circ \mathrm{RAd}\bigl(\partial^{\mu}\bigr)P. \tag{5.2}$$

In particular, if the equation $Pu = 0$ is irreducible and $\operatorname{ord} P > 1$, $\mathrm{RAd}\bigl(\partial^{-\mu}\bigr) \circ \mathrm{RAd}\bigl(\partial^{\mu}\bigr)P = cP$ with $c \in \mathbb{C}^{\times}$.

PROOF. i) Put $P = (\partial - c)Q$. Then there is a function $u(x)$ satisfying $Qu(x) = e^{cx}$. Since $Pu = 0$ has at most a regular singularity at $x = \infty$, there exist $C > 0$ and $N > 0$ such that $|u(x)| < C|x|^N$ for $|x| \gg 1$ and $0 \le \arg x \le 2\pi$, which implies $c = 0$.

ii) This follows from the fact

$$\begin{aligned}&\mathrm{Adei}\bigl(-\phi(x)\bigr) \circ \mathrm{Adei}\bigl(\phi(x)\bigr) = \mathrm{id},\\ &\mathrm{Adei}\bigl(\phi(x)\bigr)f(x)P = f(x)\,\mathrm{Adei}\bigl(\phi(x)\bigr)P \quad (f(x) \in \mathbb{C}(x))\end{aligned}$$

and the definition of $\mathrm{RAdei}\bigl(\phi(x)\bigr)$ and $\mathrm{RAd}(\partial^{\mu})$. □

The addition and the middle convolution transform the Riemann scheme of the Fuchsian differential equation as follows.

THEOREM 5.2. *Let $Pu = 0$ be a Fuchsian differential equation with the Riemann scheme* (4.15). *We assume that P has the normal form* (4.43).

i) (addition) *The operator $\mathrm{Ad}\bigl((x - c_j)^{\tau}\bigr)P$ has the Riemann scheme*

$$\begin{Bmatrix} x = c_0 = \infty & c_1 & \cdots & c_j & \cdots & c_p \\ [\lambda_{0,1} - \tau]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{j,1} + \tau]_{(m_{j,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0} - \tau]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{j,n_j} + \tau]_{(m_{j,1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{Bmatrix}.$$

ii) (middle convolution) *Fix $\mu \in \mathbb{C}$. By allowing the condition $m_{j,1} = 0$, we may assume*

$$\mu = \lambda_{0,1} - 1 \quad \text{and} \quad \lambda_{j,1} = 0 \text{ for } j = 1, \dots, p \tag{5.3}$$

and $\#\{j\,;\,m_{j,1} < n\} \ge 2$ and P is of the normal form (4.43). *Putting*

$$d := \sum_{j=0}^{p} m_{j,1} - (p-1)n, \tag{5.4}$$