Chapter 5

Monte Carlo integration

In numerical integration, we select an appropriate method in accordance with the class of functions to which the integrand in question belongs. The smoother the integrand is, the more efficient numerical integration method we can apply to it. The more complicated the integrand is, the less efficient the applicable method becomes, and the final fort is the Monte Carlo integration.

§ 5.1 deals with integrands defined on $\mathbb{T}^1$ which are $\mathcal{B}_m$-measurable for some $m \in \mathbb{N}^+$. They are univariate functions, but among them there are significant probabilistic examples to which no deterministic numerical integration methods are applicable (Example 5.8 and Example 5.9). In this section, when $2^m \gg 1$, we show that i.i.d.-sampling or pairwise independent sampling is almost optimal in the sense of what is called L^2-robustness. § 5.2 deals with important facts of RWS which we did not mention in § 2.5.2. In § 5.4, we introduce a pairwise independent sampling for all simulatable random variables (§ 1.3), which is called the *dynamical random Weyl sampling*. It is the most reliable Monte Carlo integration method as far as the author knows.

5.1 L^2-robustness

In § 2.5, we mentioned about i.i.d.-sampling and random Weyl sampling (RWS) for numerical integration of functions of m coin tosses. Here we present a special characteristic of them among all kinds of random sampling methods.

Random variables dealt with in § 2.5 are functions on $\{0,1\}^m$, which are regarded as functions on D_m, or $\mathcal{B}_m$-measurable functions on $\mathbb{T}^1$, as was stated in § 1.1. Throughout Chapter 5, integrands are, mainly, functions on $\mathbb{T}^1$.

Theorem 5.1 *(A fundamental inequality about sampling [38]) Let $\{\psi_l\}_{l=1}^{2^m-1}$ be an orthonormal system of $L^2(\mathcal{B}_m)$ such that $\int_{\mathbb{T}^1} \psi_l(x)dx = 0$ holds for each l. Then for any sequence of random variables $\{X_n\}_{n=1}^{2^m} \subset \mathbb{T}^1$, the following inequality holds.*

$$\sum_{l=1}^{2^m-1} \mathbf{E}\left[\left|\frac{1}{N}\sum_{n=1}^{N}\psi_l(X_n)\right|^2\right] \geq \frac{2^m}{N} - 1, \qquad 1 \leq N \leq 2^m. \tag{5.1}$$

Proof. Since $\psi_l \in L^2(\mathcal{B}_m)$, we may assume that $\{X_n\}_{n=1}^{2^m} \subset D_m$. As a special case, each X_n may be deterministic. Conversely, if the theorem holds for any deterministic sequence