Chapter 2

Mathematical formulation of Monte Carlo method

Formulating the Monte Carlo method as gambling, we introduce the problem of random number, which is the most essential difficulty in sampling of random variables. Then we observe basic ideas of pseudorandom generator which aims at solving the problem. In this chapter, items are treated briefly, and rigorous definitions will be introduced in the subsequent chapters.

2.1 Overview

First of all, let us show the overview of Chapter 2.

The *Monte Carlo method* is a numerical method to solve mathematical problems by computer-aided sampling of random variables. For each individual problem, we set up a probability space $(\Omega, \mathcal{F}, P)$ — in this chapter, for simplicity, we assume it to be a probability space of finite coin tosses

$$(\{0,1\}^L,\ 2^{\{0,1\}^L},\ P_L), \quad L \in \mathbb{N}^+,$$

— and a random variable S defined on it; $S : \{0,1\}^L \to \mathbb{R}$. We evaluate $S(\omega)$ by a computer for some chosen $\omega \in \{0,1\}^L$, which procedure is called *sampling*.

The Monte Carlo method is a kind of *gambling* as its name indicates. The aim of the player, say Alice, is to get a *generic value* (Figure 2.1) — a typical value or not an exceptional value — of S by sampling (§ 2.2.1). Suppose that she has no idea about which ω she should choose to get a generic value of S. She chooses an $\omega \in \{0,1\}^L$ *of her own will*, realizing the risk of getting an exceptional value of S. Her risk is measured by the probability $P_L(A)$ of a set

$$A := \left\{ \omega \in \{0,1\}^L \,\middle|\, S(\omega) \text{ is an exceptional value of } S \right\}.$$

Since S seldom takes exceptional values, we have $P_L(A) \ll 1$,[†1] and hence, Alice may well think that she will almost certainly get a generic value of S. Of course she will,

[†1] $a \ll b$ stands for "a is much less that b", while $a \gg b$ stands for "a is much greater than b".