Chapter 1

Coin tossing process

Throughout this monograph, the *coin tossing process*[†1] plays a role of the model process of random number and pseudorandom number. This may sound very restrictive for applications, but it is not. Indeed, from a coin tossing process, any practical random variables and any stochastic processes can be constructed.

1.1 Borel's model of coin tossing process

To describe m coin tosses, we use a probability space $(\{0,1\}^m, 2^{\{0,1\}^m}, P_m)$, where 0 and 1 stand for Tails and Heads respectively, and P_m stands for the uniform probability measure on $\{0,1\}^m$;

$$P_m(B) := \frac{\#B}{2^m}, \quad B \subset \{0,1\}^m \ (B \in 2^{\{0,1\}^m}).$$

But each time m changes, we must take another probability space, which is not only boring but also inconvenient when we consider limit theorems. It is a good idea to construct an infinite many coin tosses all at once on a suitable probability space. Following Borel's idea, we construct them all on the Lebesgue probability space.

Definition 1.1

1. Let $\mathbb{T}^1$ be a 1-dimensional torus, i.e., an additive group consisting of the unit interval $[0,1)$ with addition $(x+y)$ mod 1. Let $\mathcal{B}$ be a σ-algebra on $\mathbb{T}^1 = [0,1)$ consisting of all the Borel measurable sets of it, $\mathbb{P}$ be the Lebesgue measure. The triplet $(\mathbb{T}^1, \mathcal{B}, \mathbb{P})$ is called the *Lebesgue probability space*.[†2] Let $(\mathbb{T}^k, \mathcal{B}^k, \mathbb{P}^k)$ denote the k-fold direct product of $(\mathbb{T}^1, \mathcal{B}, \mathbb{P})$, which is called the *k-dimensional Lebesgue probability space*.

2. Let $d_i(x) \in \{0,1\}$ denote the i-th digit of real $x \in \mathbb{T}^1$ in its dyadic expansion;

$$x = \sum_{i=1}^{\infty} d_i(x) 2^{-i}, \quad x \in \mathbb{T}^1, \tag{1.1}$$

[†1]We call the *fair* coin tossing process simply the coin tossing process.

[†2]We sometimes consider the completion of $\mathcal{B}$ by $\mathbb{P}$, i.e., σ-algebra of all the Lebesgue measurable sets. But for numerical calculations, $\mathcal{B}$ will do.