APPENDIX

Some Properties of Invariant Polynomials

Some common materials used in this article are presented in this appendix for completeness. Most of these can be found in Kobayashi–Nomizu [50] but they are modified by following the convention in Matsushima [58]. Differences appear in coefficients, for example, $\omega \wedge \eta = \dfrac{1}{p!\,q!}\operatorname{Alt}(\omega \otimes \eta)$ for a p-form ω and a q-form η, where Alt stands for the alternizer. Another example is the formula $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$.

Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. We denote by $I^k(G)$ the set of invariant polynomials of degree k.

Definition A.1. Let $f \in I^k(G)$ and let $\varphi_1, \dots, \varphi_k$ be $\mathfrak{g}$-valued differential forms of degree $q_1, \dots, q_k$, respectively. We define a $(q_1 + \dots + q_k)$-form $f(\varphi_1, \dots, \varphi_k)$ as follows. Let $\{E_1, \dots, E_r\}$ be a basis for $\mathfrak{g}$. Then, we can write $\varphi_i = \sum_{j=1}^{r} E_j \varphi_i^j$. We set

$$f(\varphi_1, \dots, \varphi_k) = \sum_{j_1, \dots, j_k = 1}^{r} f(E_{j_1}, \dots, E_{j_k})\, \varphi_1^{j_1} \wedge \dots \wedge \varphi_k^{j_k}.$$

Notation A.2 (Chern convention). Let $f \in I^k(G)$ and let $\varphi_1, \dots, \varphi_l$ be $\mathfrak{g}$-valued differential forms as above. If $l < k$, then we set

$$f(\varphi_1, \dots, \varphi_l) = f(\varphi_1, \dots, \varphi_{l-1}, \overbrace{\varphi_l, \dots, \varphi_l}^{k-l+1 \text{ times}}).$$

Definition A.3. Let $f \colon \mathfrak{gl}(n; C) \to \mathbb{C}$ be a multilinear mapping invariant under the adjoint action. The polarization of f is the unique element $\widehat{f}$ of $I^k(\mathrm{GL}(n; \mathbb{C}))$ such that

$$\widehat{f}(X, X, \dots, X) = f(X)$$