CHAPTER 2

Relation between Real and Complex Secondary Classes

A natural mapping $\varphi\colon B\Gamma_q^{\mathbb{C}} \to B\Gamma_{2q}$ is obtained by forgetting transverse complex structures. There is a natural homomorphism from $H^*(\mathrm{WO}_{2q})$ to $H^*(\mathrm{WU}_q)$ which corresponds to this mapping as follows.

THEOREM 2.1 ([64], [3, Theorem 3.1]). *Let λ be the mapping from* WO_{2q} *to* WU_q *given by*

$$\lambda(c_k) = (\sqrt{-1})^k \sum_{j=0}^{k} (-1)^j v_{k-j}\overline{v}_j,$$

$$\lambda(h_{2k+1}) = \frac{(-1)^k}{2}\sqrt{-1}\sum_{j=0}^{2k+1}(-1)^j \widetilde{u}_{2k-j+1}(v_j + \overline{v}_j),$$

where v_0 and $\overline{v}_0$ are considered as 1. Then λ induces a homomorphism from $H^*(\mathrm{WO}_{2q})$ to $H^*(\mathrm{WU}_q)$, denoted by $[\lambda]$. The homomorphism $[\lambda]$ corresponds to forgetting transverse complex structures, indeed, the following diagram commutes:

$$\begin{array}{ccc} H^*(\mathrm{WO}_{2q}) & \xrightarrow{[\lambda]} & H^*(\mathrm{WU}_q) \\ \chi\downarrow & & \downarrow\chi^{\mathbb{C}} \\ H^*(B\Gamma_{2q}) & \xrightarrow[\varphi^*]{} & H^*(B\Gamma_q^{\mathbb{C}}). \end{array}$$

The Godbillon–Vey class and the imaginary part of the Bott class are related by the formula

$$[\lambda](\mathrm{GV}_{2q}) = \frac{(2q)!}{q!\,q!}\,\xi_q \cdot \mathrm{ch}_1^q\,,$$