Chapter 6

Decay of solutions to the Cauchy problem

Recall that we begin with the Cauchy problem with solution $u = u(t, x)$

$$\begin{cases} D_t^m u + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0, \quad t > 0, \\ D_t^l u(0, x) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l = 0, \dots, m-1,\ x \in \mathbb{R}^n\,, \end{cases} \tag{6.0.1}$$

where $P_j(\xi)$, the polynomial obtained from the operator $P_j(D_x)$ by replacing each derivative $D_{x_k} = \frac{1}{i}\partial_{x_k}$ by ξ_k, is a constant coefficient homogeneous polynomial of order j, and the $c_{\alpha,r}$ are constants. In this section we will prove different parts of Theorem 2.4.1.

6.1 Representation of the solution

Applying the partial Fourier transform with respect to x yields an ordinary differential equation for $\widehat{u} = \widehat{u}(t, \xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(t, x)\, dx$:

$$D_t^m \widehat{u} + \sum_{j=1}^{m} P_j(\xi) D_t^{m-j} \widehat{u} + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} \xi^\alpha D_t^r \widehat{u} = 0\,, \tag{6.1.1a}$$

$$D_t^l \widehat{u}(0, \xi) = \widehat{f_l}(\xi), \quad l = 0, \dots, m-1, \tag{6.1.1b}$$

where $(t, \xi) \in [0, \infty) \times \mathbb{R}^n$ and $P_j(\xi)$ are symbols of $P_j(D_x)$. Let $E_j = E_j(t, \xi)$, $j = 0, \dots, m-1$, be the solutions to (6.1.1a) with initial data

$$D_t^l E_j(0, \xi) = \begin{cases} 1 & \text{if } l = j, \\ 0 & \text{if } l \neq j. \end{cases} \tag{6.1.1c}$$