Chapter 2

Main estimates

We will now turn to analysing the conditions under which we can obtain $L^p - L^q$ decay estimates for the general m^{th} order linear, constant coefficient, strictly hyperbolic Cauchy problem

$$\begin{cases} L(D_t, D_x) \equiv D_t^m u + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0, \ t > 0, \\ D_t^l u(0,x) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l = 0, \dots, m-1, \ x \in \mathbb{R}^n \,. \end{cases} \tag{2.0.1}$$

Results of this section will show how different behaviours of the characteristic roots $\tau_1(\xi), \dots, \tau_m(\xi)$ affect the rate of decay that can be obtained. As in the introduction, the symbol $P_j(\xi)$ of $P_j(D_x)$ is a homogeneous polynomial of order j, and the $c_{\alpha,r}$ are constants. The differential operator in the first line of (2.0.1) will be denoted by $L(D_t, D_x)$ and its symbol by $L(\tau, \xi)$. The principal part of L is denoted by L_m. Thus, $L_m(\tau, \xi)$ is a homogeneous polynomial of order m. In the subsequent analysis, ideally, of course, we would like to have conditions on the lower order terms for different rates of decay; in Section 8 we shall give some results in this direction. For now, though, we concentrate on conditions on the characteristic roots.

First of all, it is natural to impose the stability condition, namely that for all $\xi \in \mathbb{R}^n$ we have

$$\operatorname{Im} \tau_k(\xi) \geq 0 \quad \text{for } k = 1, \dots, m\,; \tag{2.0.2}$$

this is equivalent to requiring the characteristic polynomial of the operator to be stable at all points $\xi \in \mathbb{R}^n$, and thus cannot be expected to be lifted. In fact, certain microlocal decay estimates are possible even without this condition if the supports of the Fourier transforms of the Cauchy data are contained in the set where condition (2.0.2) holds. However, this restriction