Part VIII
Hypergeometric constructions of rational approximations for (multiple) zeta values

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This survey presents certain results concerning the diophantine nature of zeta values or multiple zeta values that I have obtained over the last few years, with or without coauthors. I will not try to cover all the known results concerning the diophantine theory of the Riemann zeta function and more information is available in [12] for example. The first part is a presentation of irrationality results for the values of the Riemann zeta function, together with a description of the memoir [17] joint with Christian Krattenthaler on the "Denominators conjecture". The second part describes some of the results in two papers with J. Cresson and S. Fischler [10, 11], both devoted to the construction of linear forms in multiple zeta values, which are generalisations of Riemann zeta function. I warmly thank K. Matsumoto and H. Tsumura, the organisers of the franco-japanese winter school in January 2008 at Miura seaside, for giving me the opportunity to publish this survey in the present lecture notes.

In diophantine approximation, proofs of irrationality, linear independence, etc, usually rely on the construction of "auxiliary functions" and this is also the case here. Indeed, the results presented in both parts of the paper are proved by means of explicit auxiliary functions, which turn out to be hypergeometric series in one or several variables. (The underlying aspect "Padé approximants" will not be developped.) Therefore, before going into the subject, it is useful to remind the reader of the definition of those series. These are power series defined by

$$ {}_{q+1}F_q\left[\begin{matrix}\alpha_0, \alpha_1, \ldots, \alpha_q \\ \beta_1, \ldots, \beta_q\end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(\alpha_0)_k\,(\alpha_1)_k \cdots (\alpha_q)_k}{k!\,(\beta_1)_k \cdots (\beta_q)_k} z^k, $$

where $\alpha_j \in \mathbb{C}$, $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and $(x)_m = x(x+1)\cdots(x+m-1)$ is the Pochhammer symbol. Such series converge for all $z \in \mathbb{C}$ such that $|z| < 1$, and for $z = \pm 1$, provided that $\mathrm{Re}(\beta_1 + \cdots + \beta_q) > \mathrm{Re}(\alpha_0 + \alpha_1 + \cdots + \alpha_q)$. The literature (see [3, 16, 22]) contains various special kind of hypergeometric series whose parameters satisfy particular relations. For example, a hypergeometric series is said to be

- *balanced* if $\alpha_0 + \cdots + \alpha_q + 1 = \beta_1 + \cdots + \beta_q$;
- *nearly-poised (of the first kind)* if $\alpha_1 + \beta_1 = \cdots = \alpha_q + \beta_q$;
- *well-poised* if $\alpha_0 + 1 = \alpha_1 + \beta_1 = \cdots = \alpha_q + \beta_q$;
- *very-well-poised* if it is well-poised and $\alpha_1 = \frac{1}{2}\alpha_0 + 1$.

In ths first section, we will show that the very-well-poised case is of special importance. In the second section, we will present multiple series which are non-trivial generalisation of one variable very-well-poised series.