Part V

Rankin-Selberg method and periods of modular forms

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0 Introduction

Let $f(z) = \sum_{m=1}^{\infty} a(m)\exp(2\pi imz)$ and $g(z) = \sum_{m=1}^{\infty} b(m)\exp(2\pi imz)$ be cusp forms for $SL_2(\mathbf{Z})$. Then the Rankin-Selberg method gives an integral representation of the Dirichlet series, called the Rankin-Selberg convolution product, defined by

$$L(s,f,g) = \sum_{m=1}^{\infty} a(m)\overline{b(m)}m^{-s}.$$

This method was first introduced by Rankin [27] and Selberg [29] independently. Since then, it has fully developed for several types of modular forms, and has become one of the most useful tools for studying modular forms and their L-functions. In particular, it plays a very important role in proving analytic properties (meromorphy, functional equation etc.) of several automorphic L-functions. As for this, the reader is referred to excellent surveys by Bump [2] and [3].

In this paper, we give another application of the Rankin-Selberg method, which expresses the period of a cuspidal Hecke eigenform in terms of the special values of automorphic L-functions related to it. Here we mean by the period of a cusp form f the Petersson product $\langle f, f\rangle$ of f in almost all cases. The main purposes of this paper are as follows:

(1) to survey Petersson's formula for the period of an elliptic cusp form and its application;
(2) to survey Kohnen-Zagier's formula for the period of a Hecke eigenform of half integral weight;
(3) to give an outline of the proof of Ikeda's conjecture on the period of the Ikeda lift.

To explain them more precisely, first let f be a cusp form of integral weight k for $\Gamma_0(N)$. Then, in Section 2, we give Petersson's formula, which expresses the period $\langle f, f\rangle$ in terms of the residue of the Rankin-Selberg convolution product $L(s,f,f)$ at $s = k$. (cf. Proposition 2.3.) This is due to Petersson [26]. As an application, we express $\langle f, f\rangle$ in terms of the adjoint L-function of f evaluated at $s = 1$ in case f is a normalized Hecke eigenform (cf. Theorem 2.4.) This topic is rather elementary and well-known but instructive for our later investigation. So I will explain it precisely. Furthermore, we consider the algebraicity of the special values of several L-functions.

Next let f be a Hecke eigenform in the Kohnen plus subspace of cusp forms of weight $k + 1/2$ for $\Gamma_0(4)$, and $S(f)$ the normalized Hecke eigenform of weight $2k$ for $SL_2(\mathbf{Z})$ corresponding to f under the Shimura correspondence. Then, in Section 3, we explain Kohnen-Zagier's formula, which expresses the ratio $\dfrac{\langle S(f), S(f)\rangle}{\langle f, f\rangle}$ of the periods in terms of the Fourier coefficient of f and the central critical value of the twisted Hecke L-function of $S(f)$ (cf. Theorem 3.4.) This type of result