# Part III
# Spherical functions on $p$-adic homogeneous spaces

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## Introduction

Let $\mathbb{G}$ be a reductive linear algebraic group defined over $k$, and $\mathbb{X}$ be an affine algebraic variety defined over $k$ which is $\mathbb{G}$-homogeneous, where and henceforth $k$ stands for a non-archimedian local field of characteristic 0. The Hecke algebra $\mathcal{H}(G, K)$ of $G$ with respect to $K$ acts by convolution product on the space of $\mathcal{C}^\infty(K\backslash X)$ of $K$-invariant $\mathbb{C}$-valued functions on $X$, where $K$ is a maximal compact open subgroup of $G = \mathbb{G}(k)$ and $X = \mathbb{X}(k)$.

A nonzero function in $\mathcal{C}^\infty(K\backslash X)$ is called *a spherical function on* $X$ if it is a common $\mathcal{H}(G, K)$-eigen function.

Spherical functions on homogeneous spaces comprise an interesting topic to investigate and a basic tool to study harmonic analysis on $G$-space $X$. They have been studied also as spherical vectors of distinguished models, Shalika functions and Whittaker-Shintani functions, and are closely related to theory of automorphic forms and representation theory. When $\mathbb{G}$ and $\mathbb{X}$ are defined over $\mathbb{Q}$, spherical functions appear in local factors of global objects, e.g. Rankin-Selberg convolutions and Eisenstein series (e.g. [CS], [Fl], [HS3], [Jac], [KMS], [Sf2]).

The theory of spherical functions also has applications to classical number theory. For example when $X$ is the space of symmetric forms, alternating forms or hermitian forms, spherical functions can be considered as generating functions of local densities, and have been applied to obtain their explicit formulas (cf. [HS1], [HS2], [H1]-[H4]).

To obtain explicit expressions of spherical functions is one of basic problems. For the group cases, it has been done by I. G. Macdonald and afterwards by W. Casselman by a representation theoretical method (cf. [Ma], [Cas]). There are some results on homogeneous space cases mainly for the case that the space of spherical functions attached to each Satake parameter is of dimension one (e.g. [CS], [KMS], [Of]).

In this paper, following the preliminaries in §1, we give a general expression of spherical functions on $X$ of dimension not necessary one based on the data of the group $G$ and functional equations of spherical functions in §2. Then we show a unified method to obtain functional equations of spherical functions on $X$, and explain that functional equations are reduced to those of $p$-adic local zeta functions of *small* prehomogeneous vector spaces in §3. These are improvements of some results in [H3] and [H6]. We devote §4 to examples. For general references for algebraic groups, one may refer to [Bo] and [PR].

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