# Part II
# Lectures on height zeta functions: At the confluence of algebraic geometry, algebraic number theory, and analysis

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## 1 Introduction

### 1.1 Diophantine equations and geometry

#### 1.1.1 Diophantine equations

Broadly speaking, *arithmetic* is the study of diophantine equations, that is, systems of polynomial equations with *integral* coefficients, with a special emphasis on their solutions in rational integers. Of course, there are numerous variants, the most obvious ones allowing to consider coefficients and solutions in the field of rational numbers, or in more general number fields, or even in more general fields, *e.g.*, finite fields.

The reader should be warned that, in this generality, we are constrained by the *undecidability theorem* of [32]: there is no general method, that is no algorithm, to decide whether or not any given polynomial system has solutions in rational integers. Any mathematician working on diophantine equations is therefore obliged to consider specific types of diophantine equations, in the hope that such an undecidability issues do not apply within the chosen families of equations.

#### 1.1.2 Enters geometry

At first, one is tempted to sort equations according to the degrees of the polynomials which intervene. However, this approach is much too crude, and during the XXth century, mathematicians were led to realize that there are profound relations between the given diophantine equation and the *geometry* of its solutions in real or complex numbers. This led to considerations of geometric invariants such as the *genus* of an algebraic curve, to the (essentially opposite) notions of a variety of general type and a Fano variety, to the notion of a rational variety, etc.

In this survey, we are interested in diophantine equations having infinitely many solutions. A natural way to describe this infinite set consists in sorting the solutions according to their size (as integers) and in studying the asymptotic behaviour of the number of solutions of size smaller than a growing bound.

#### 1.1.3 The circle problem

The classical *circle problem* in analytic number theory is to estimate the number of integer vectors $\mathbf{x} \in \mathbf{Z}^n$ such that $s(\mathbf{x}) \leqslant B$, when $B \to \infty$ and $s(\cdot)$ is an appropriate notion of a size of a vector in $\mathbf{R}^n$. When $s(\cdot) = \|\cdot\|$ is a norm, for example the euclidean norm, this amounts to counting the number of lattice points in a ball with center 0 and of radius $B$.