Part I
Analytic continuation of some zeta functions

Gautami Bhowmik

0 Introduction

The contents of this paper were presented as lectures at the Miura Winter School on Zeta and L-functions held in 2008. Though the analytic continuation of zeta functions beyond its region of absolute convergence is a fundamental question, in general not much is known about the conditions that guarantee a meromorphic continuation. It is also interesting to know how far such a function can be continued, that is where the natural boundary of analytic continuation lies.

The choice of functions that are considered here are 'arbitrary', that is a matter of personal taste and expertise. Most of the work reported is on what I have studied or actually contributed to together with my co-authors. The word 'some' in the title is to indicate that though the paper is expository, it is not exhaustive. Only outlines of proofs have sometimes been provided.

In the first part we consider Euler products. One of the most important applications of zeta functions is the asymptotic estimation of the sum of its coefficients via Perron's formula, that is, the use of the equation

$$\sum_{n\leq x} a_n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Big(\sum_{n\geq 1}\frac{a_n}{n^s}\Big)\frac{x^s}{s}\,ds.$$

To use this relation, one usually shifts the path of integration to the left, thereby reducing the contribution of the term x^s. This becomes possible only if the function $D(s)=\sum \frac{a_n}{n^s}$ is holomorphic on the new path. In Section 3 details of certain examples from height zeta functions and zeta functions of groups have been given.

Clearly all zeta functions do not have Euler product expansions, one important class of examples being multiple zeta functions which have been studied often in recent years. Not many general methods exist and here I treat the case of the Goldbach generating function associated to $G_r(n)$, the number of representations of n as the sum of r primes

$$\sum_{k_1=1}^{\infty}\cdots\sum_{k_r=1}^{\infty}\frac{\Lambda(k_1)\ldots\Lambda(k_r)}{(k_1+k_2+\cdots+k_r)^s}=\sum_{n=1}^{\infty}\frac{G_r(n)}{n^s}.$$

where Λ is the classical von-Mangoldt function.

In almost all examples the natural boundary, if it can be obtained, corresponds to the intuitively expected boundary and this can in fact be proved in a probabilistic sense. However one of the difficulties in actually obtaining the boundary is that our analyses often depend on the distribution of zeros of the Riemann zeta function, and thus on yet unproved hypotheses (see, for example, Theorem 1.3 or Theorem 3.4 below).

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