CHAPTER 6

Combable functions and ergodic theory

In this chapter we study quasimorphisms on hyperbolic groups, especially counting quasimorphisms, from a *computational* perspective. We introduce the class of *combable functions* (and the related classes of weakly combable and bicombable functions) on a hyperbolic group, and show that the Epstein–Fujiwara counting functions are bicombable.

Conversely, bicombable function satisfying certain natural conditions are shown to be quasimorphisms; thus quasimorphisms and bounded cohomology arise naturally in the study of automatic structures on hyperbolic groups, a fact which might at first glance seem surprising.

The (asymptotic) distribution of values of a combable function may be described very simply using stationary Markov chains. Consequently, we are able to derive a central limit theorem for the distribution of values of counting quasimorphisms on hyperbolic groups.

The main reference for this section is Calegari–Fujiwara [**50**], although Picaud [**166**] and Horsham–Sharp [**113**] are also relevant.

6.1. An example

6.1.1. Random walk on $\mathbb{Z}$.

Definition 6.1. A sequence of integers $x = (x_0, x_1, \cdots)$ is a *walk* on $\mathbb{Z}$ if it satisfies the following two properties:

(1) (initialization) $x_0 = 0$
(2) (unit step) for all $n > 0$, there is an equality $|x_n - x_{n-1}| = 1$

The *length* of a walk x is *one less* than the number of terms in the sequence x. So, for example, $(0, 1, 2)$ has length 2, while $(0, 1, 0, -1, -2)$ has length 4.

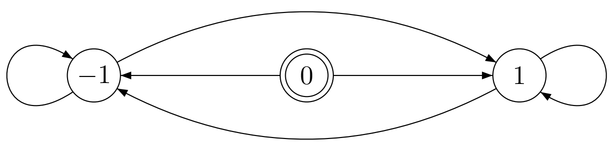

Figure 6.1. Walks on $\mathbb{Z}$ of length n are in bijection with walks on Γ of length n.

Knowing the successive differences $x_n - x_{n-1} \in \{-1, 1\}$ determines x, so there is a bijection between walks of length n, and strings of length n in the alphabet