CHAPTER 5

Irrationality and dynamics

The set of values of scl on all conjugacy classes in all finitely presented groups is a countable set. It is natural to try to characterize this set of real numbers, and to understand what kinds of arithmetic constraints exist on the values of scl in certain classes of groups.

As discussed in Chapter 4, the Rationality Theorem (i.e. Theorem 4.24) shows that for free groups (and more generally, for PQL groups) the scl norm is rational, and in particular, scl takes on values in $\mathbb{Q}$ in free groups. More generally, we saw that the unit ball of the scl norm on $B_1^H(F)$ is a rational polyhedron, and discussed the relationship of this example to the (polyhedral) Thurston norm on H_2 of an atoroidal irreducible 3-manifold.

It is natural to ask for which groups G the stable commutator length is rational on $[G, G]$. In fact, Gromov ([**99**], 6.C) explicitly asked whether scl is always rational, or at least algebraic, in general finitely presented groups. In the next section we describe an unexpected and elegant example due to Dongping Zhuang [**205**] of a finitely presented group in which the stable commutator length achieves *transcendental values*, thus answering Gromov's question in the negative.

There are two essential ingredients in Zhuang's examples: the groups he considers are *transformation groups* (i.e. groups of automorphisms of some geometric object), and they have an *arithmetic* origin. It is a general phenomenon, observed explicitly by Burger–Monod, Carter–Keller–Paige (as exposed by Dave Witte-Morris) and others, that (especially arithmetic) lattices in higher rank Lie groups generally admit no (nontrivial) quasimorphisms. On the other hand, such groups sometimes have nontrivial 2-dimensional bounded cohomology classes, which typically have a symplectic (or "causal") origin, which can be detected dynamically by realizing the groups as transformation groups. A central extension of such a group admits a nontrivial, but *finite dimensional* space of homogeneous quasimorphisms, and one may compute scl on such a group directly by Bavard duality, relating scl to dynamics.

In § 5.1 we discuss Zhuang's examples, which in some ways are the most elementary. In § 5.2 we discuss lattices in higher rank Lie groups from several different perspectives, eventually concentrating on lattices in symplectic groups as the most interesting examples. Finally, in § 5.3, we discuss some nonlinear generalizations of these ideas, which leads to the construction of quasimorphisms on braid groups and certain (low-dimensional) groups of area-preserving diffeomorphisms of surfaces. References for this chapter include [**28, 192, 205, 33, 34, 53, 159, 7, 86, 87**].

5.1. Stein–Thompson groups

In 1965, Richard Thompson [**195**] defined three groups $F \subset T \subset V$. Two of these (the groups T and V) were the first examples of finitely-presented, infinite simple groups. They can be defined as transformation groups (i.e. as groups of