CHAPTER 4

Free and surface groups

In this chapter we study scl in free groups, and some related groups. The methods are largely geometric and depend on realizing the groups in question as fundamental groups of particularly simple low-dimensional manifolds.

The first main theorem proved in this chapter is the Rationality Theorem (Theorem 4.24), which says that in a free group F, the unit ball of the scl norm on $B_1^H(F)$ is a rational polyhedron; i.e. scl is a piecewise linear rational function on finite dimensional rational subspaces of $B_1^H(F)$. It follows that scl takes on only rational values in free groups. The method of proof is direct: we show how to explicitly construct extremal surfaces bounding finite linear combinations of conjugacy classes. As a byproduct, we obtain a polynomial-time algorithm to calculate scl in free groups, which can be practically implemented, at least in some simple cases. This algorithm gives an interesting conjectural picture of the spectrum of scl on free groups, and perhaps some insight into the spectrum of scl on word-hyperbolic groups in general.

The polyhedrality of the unit ball of the scl norm is related to certain rigidity phenomena. Each nonzero element in $B_1^H(F)$ projectively intersects the boundary of the unit ball of the scl norm in the interior of some face. The smaller the codimension of this face, the smaller the space of quasimorphisms which are extremal for the given element. The situations displaying the most rigidity are therefore associated to faces of the unit ball of codimension one. It turns out that for a free group, such faces of codimension one exist, and have a geometric meaning. In § 4.2 we discuss the Rigidity Theorem (Theorem 4.78), which says that if F is a free group, associated to each isomorphism $F \to \pi_1(S)$ (up to conjugacy), where S is a compact oriented surface, there is a top dimensional face π_S of the unit ball of the scl norm on F, and the unique homogeneous quasimorphism ϕ_S dual to π_S (up to scale and elements of H^1) is the rotation quasimorphism associated to a hyperbolic structure on S.

Finally, in § 4.3, we discuss diagrammatic methods to study scl in free groups. In particular, we discuss a technique due to Duncan–Howie which uses left-invariant orders on one-relator groups to obtain sharp lower bounds on scl in free groups.

Some of the material in this chapter is developed more fully in the papers [**47, 43, 45, 46**].

4.1. The Rationality Theorem

The goal of this section is to prove the Rationality Theorem for free groups. Essentially, this theorem says that the unit ball in the scl norm on $B_1^H(F)$ is a rational polyhedron. Polyhedral norms occur in other contexts in low-dimensional