CHAPTER 3

Hyperbolicity and spectral gaps

There are two main sources of quasimorphisms: *hyperbolic geometry* (i.e. negative curvature) and *symplectic geometry* (i.e. partial orders and causal structures). In this chapter we study scl in hyperbolic manifolds, and more generally, in word-hyperbolic groups in the sense of Rips and Gromov [**98**] and groups acting on hyperbolic spaces (we return to symplectic geometry, and quasimorphisms with a dynamical or causal origin in Chapter 5). The construction of explicit quasimorphisms is systematized by Bestvina–Fujiwara ([**13**]), who show that in order to construct (many) quasimorphisms on a group G, it suffices to exhibit an isometric action of G on a δ-hyperbolic space X which is *weakly properly discontinuous* (see Definition 3.51). It is crucial for many important applications that X need not be itself proper.

The relationship between negative curvature and quasimorphisms is already evident in the examples from § 2.3.1. If M is a closed hyperbolic manifold, the space of smooth 1-forms $\Omega^1 M$ injects into $Q(\pi_1(M))$. Evidently, quasimorphisms are sensitive to a great deal of the geometry of M; one of the goals of this chapter is to sharpen this statement, and to say what kind of geometry quasimorphisms are sensitive to.

A fundamental feature of the geometry of hyperbolic manifolds is the *thick-thin decomposition*. In each dimension n there is a universal constant $\epsilon(n)$ (the *Margulis constant*) such that the part of a hyperbolic n-manifold M with injectivity radius less than ϵ (i.e. the "thin" piece) has very simple topology — each component is either a neighborhood of a cusp, or a tubular neighborhood of a single short embedded geodesic. Margulis' observation implies that in each dimension, there is a *universal* notion of what it means for a closed geodesic to be *short*.

In this chapter we prove fundamental inequalities relating length to scl in hyperbolic spaces and to show that there is a *universal* notion of what it means for a conjugacy class in a hyperbolic group to have small scl. We think of this as a kind of *homological Margulis Lemma*. These inequalities generalize to certain groups acting on hyperbolic spaces, such as amalgamated free products and mapping class groups of surfaces.

Much of the content in this chapter is drawn from papers of Bestvina, Calegari, Feighn, and Fujiwara (sometimes in combination), especially [**82, 83, 13, 42, 49, 12**].

3.1. Hyperbolic manifolds

We start with the simplest and most explicit examples of groups acting on hyperbolic spaces, namely fundamental groups of hyperbolic manifolds. In this context, scl can be controlled by directly studying maps of surfaces to manifolds.