CHAPTER 1

Surfaces

In this chapter we present some of the elements of the geometric theory of 2-dimensional (bounded) homology in an informal way. The main purpose of this chapter is to standardize definitions, to refresh the reader's mind about the relationship between 2-dimensional homology classes and maps of surfaces, and to compute the Gromov norm of a hyperbolic surface with boundary. All of this material is essentially elementary and many expositions are available; for example, [**10**] covers this material well.

We start off by discussing maps of surfaces into topological spaces. One way to study such maps is with linear algebra; this way leads to homology. The other way to study such maps is with group theory; this way leads to the fundamental group and the commutator calculus. These points of view are reconciled by Hopf's formula; a more systematic pursuit leads to rational homotopy theory.

1.1. Triangulating surfaces

A *surface* is a topological space (usually Hausdorff and paracompact) which is locally two dimensional. That is, every point has a neighborhood which is homeomorphic to the plane, usually denoted by $\mathbb{R}^2$.

1.1.1. The plane. It is unfortunate in some ways that the standard way to refer to the plane emphasizes its product structure. This product structure is topologically unnatural, since it is defined in a way which breaks the natural topological symmetries of the object in question. This fact is thrown more sharply into focus when one discusses more rigid topologies.

EXAMPLE 1.1 (Zariski topology). The product topology on two copies of the affine line with its Zariski topology is *not* typically the same as the Zariski topology on the affine plane. A closed set in $\mathbb{R}^1$ with the Zariski topology is either all of $\mathbb{R}$, or a finite collection of points. A closed set in $\mathbb{R}^2$ with the product topology is therefore either all of $\mathbb{R}^2$, or a finite union of horizontal and vertical lines and isolated points. By contrast, closed sets in the Zariski topology in $\mathbb{R}^2$ include circles, ovals, and algebraic curves of every degree.

Part of the bias is biological in origin:

EXAMPLE 1.2 (Primary visual cortex). The primary visual cortex of mammals (including humans), located at the posterior pole of the occipital cortex, contains neurons hardwired to fire when exposed to certain spatial and temporal patterns. Certain specific neurons are sensitive to stimulus along specific orientations, but in primates, more cortical machinery is devoted to representing vertical and horizontal than oblique orientations (see for example [**58**] for a discussion of this effect).