Chapter 4. An analogue of W for discrete Markov chains.

4.0 Introduction.

In this chapter, we construct for Markov chains some σ-finite measures which enjoy similar properties as the measure **W** studied in Chapter 1. Very informally, these σ-finite measures are obtained by "conditioning a recurrent Markov process to be transient".

Our construction applies to discrete versions of one- and two-dimensional Brownian motion, i.e. simple random walk on $\mathbb{Z}$ and $\mathbb{Z}^2$, but it can also be applied to a much larger class of Markov chains.

This chapter is divided into three sections; in Section 4.1, we give the construction of the σ-finite measures mentioned above ; in Section 4.2, we study the main properties of these measures, and in Section 4.3, we study some examples in more details.

4.1 Construction of the σ-finite measures $(\mathbb{Q}_x, x \in E)$

4.1.1 Notation and hypothesis.

Let E be a countable set, $(X_n)_{n\geq 0}$ the canonical process on $E^{\mathbb{N}}$, $(\mathcal{F}_n)_{n\geq 0}$ its natural filtration, and $\mathcal{F}_\infty$ the σ-field generated by $(X_n)_{n\geq 0}$.
Let us denote by $(\mathbb{P}_x)_{x\in E}$ the family of probability measures on $(E^{\mathbb{N}}, (\mathcal{F}_n)_{n\geq 0}, \mathcal{F}_\infty)$ associated to a Markov chain ($\mathbb{E}_x$ below denotes the expectation with respect to $\mathbb{P}_x$) ; more precisely, we suppose there exist probability transitions $(p_{y,z})_{y,z\in E}$ such that :

$$\mathbb{P}_x(X_0 = x_0, X_1 = x_1, ..., X_k = x_k) = \mathbf{1}_{x_0=x} p_{x_0,x_1} p_{x_1,x_2} ... p_{x_{k-1},x_k} \tag{4.1.1}$$

for all $k \geq 0$, $x_0, x_1, ..., x_k \in E$.

We assume three more hypotheses :

- For all $x \in E$, the set of $y \in E$ such that $p_{x,y} > 0$ is finite (i.e. the graph associated to the Markov chain is locally finite).
- For all $x, y \in E$, there exists $n \in \mathbb{N}$ such that $\mathbb{P}_x(X_n = y) > 0$ (i.e. the graph of the Markov chain is connected).
- For all $x \in E$, the canonical process is recurrent under the probability $\mathbb{P}_x$.

4.1.2 A family of new measures.

From the family of probabilities $(\mathbb{P}_x)_{x\in E}$, we will construct families of σ-finite measures which should be informally considered to be the law of $(X_n)_{n\geq 0}$ under $\mathbb{P}_x$, after conditionning this process to be transient.
More precisely, let us fix a point $x_0 \in E$ and let us suppose there exists a function $\phi : E \to \mathbb{R}_+$ such that :

- $\phi(x) \geq 0$ for all $x \in E$, and $\phi(x_0) = 0$.
- ϕ is harmonic with respect to $\mathbb{P}$, except at the point x_0, i.e. :
  for all $x \neq x_0$, $\sum_{y\in E} p_{x,y}\phi(y) = \mathbb{E}_x[\phi(X_1)] = \phi(x)$.
- ϕ is unbounded.