Chapter 3. The analogue of the measure W for a class of linear diffusions.

In Chapters 1 and 2, we have, starting from penalisation results, associated to Wiener measure in dimensions 1 and 2 a positive and σ-finite measure $\mathbf{W}$ (resp. : $\mathbf{W}^{(2)}$ in dimension 2) on the canonical space $(\Omega, \mathcal{F}_\infty)$. In this 3rd Chapter, we shall prove the existence of a measure which is analogous to $\mathbf{W}$, in the more general situation of a large class of linear diffusions. This class is described in Section 3.2. Our approach in this Chapter does not use any penalisation result. Then, in Section 3.3, we shall particularize these results about linear diffusions to the situation of Bessel processes with dimension $d = 2(1-\alpha)$ $(0 < d < 2$, or $0 < \alpha < 1)$. Thus, we shall obtain the existence of the measure $\mathbf{W}^{(-\alpha)}(0 < \alpha < 1)$ on $\big(\mathcal{C}(\mathbb{R} \to \mathbb{R}_+),\ \mathcal{F}_\infty\big)$ and we shall then indicate its relationship with penalisation problems. Section 3.1 is devoted to a presentation of our hypotheses and notations.

3.1 Main hypotheses and notations.

3.1.1 Our framework is that of Salminen-Vallois-Yor. [SVY], that is :
$(X_t,\ t \geq 0)$ is a $\mathbb{R}_+ = [0, \infty[$ valued diffusion, with 0 an instantaneously reflecting barrier. The infinitesimal generator $\mathcal{G}$ of $(X_t,\ t \geq 0)$ is given by :

$$\mathcal{G}\, f(x) = \frac{d}{dm}\,\frac{d}{dS}\, f(x) \qquad (x \geq 0) \tag{3.1.1}$$

where the scale function S is a continuous, strictly increasing function s.t. :

$$S(0) = 0, \quad S(+\infty) = +\infty \tag{3.1.2}$$

and $m(dx)$ is the speed measure of X ; we assume $m(\{0\}) = 0$.

3.1.2 The semi-group of $(X_t,\ t \geq 0)$ admits $p(t,x,y)$ as density with respect to m :

$$P_x(X_t \in dy) = p(t,x,y)m(dy) \tag{3.1.3}$$

with p continuous in the 3 variables, and $p(t,x,y) = p(t,y,x)$. $\widehat{X}$ denotes the process X, killed at $T_0 = \inf\{t \ ;\ X_t = 0\}$. We denote by $\widehat{p}$ its density with respect to m :

$$\widehat{P}_x(\widehat{X}_t \in dy) = P_x(X_t \in dy \ ;\ 1_{t<T_0}) := \widehat{p}(t,x,y)m(dy) \tag{3.1.4}$$

with $\widehat{p}(t,x,y) = p(t,x,y)P_x(T_0 > t | X_t = y)$.

3.1.3 The local time process
We denote by $\{L_t^y \ ;\ t \geq 0,\ y \geq 0\}$ the jointly continuous family of local times of X, which satisfy the density of occupation formula :

$$\int_0^t h(X_s)ds = \int_0^\infty h(y) L_t^y\, m(dy) \tag{3.1.5}$$

for any $h : \mathbb{R}_+ \to \mathbb{R}_+$, Borel. It is easily deduced from (3.1.5) and (3.1.3) that :

$$E_x(d_t\, L_t^y) = p(t,x,y)dt \tag{3.1.6}$$

We denote by $P_0^{\tau_l}$ the law, under P_0, of $(X_t,\ t \leq \tau_l)$ with $\tau_l := \inf\{t \geq 0 \ ;\ L_t^0 > l\}$. We have also :

$$\big(S(X_t) - L_t,\ t \geq 0\big) \quad \text{is a martingale} \tag{3.1.7}$$