Chapter 2. Existence and properties of the measure $\mathbf{W}^{(2)}$.

We shall now establish a number of results similar to those of Chapter 1, but this time $(X_t,\ t \geq 0)$ is a 2-dimensional Brownian motion.

2.1. Existence of $\mathbf{W}^{(2)}$.

2.1.1 Notations and Feynman-Kac penalisations in two dimensions.

$\big(\Omega = \mathcal{C}(\mathbb{R}_+ \to \mathbb{C}), (X_t,\ \mathcal{F}_t)_{t\geq 0},\ W_z^{(2)}(z \in \mathbb{C})\big)$ denotes the two dimensional canonical Brownian motion, which takes its values in $\mathbb{C}$. We write $W^{(2)}$ for $W_0^{(2)}$. $\mathcal{I}$ denotes here the set of positive Radon measures on $\mathbb{C}$ admitting a density q with compact support and such that $\displaystyle\int q(x)dx > 0$. Define :

$$A_t^{(q)} := \int_0^t q(X_s)ds \tag{2.1.1}$$

Here is the analogue in dimension 2 of Theorem 1.1.1. A proof of this Theorem (in dimension 2) is found in [RVY, VI].

Theorem 2.1.1. *Let $q \in \mathcal{I}$ and, for every $t \geq 0$ and $z \in \mathbb{C}$:*

$$W_{z,t}^{(2,q)} := \frac{\exp\big(-\frac{1}{2}A_t^{(q)}\big)}{Z_{z,t}^{(2,q)}} \cdot W_z^{(2)} \tag{2.1.2}$$

with

$$Z_{z,t}^{(2,q)} := W_z^{(2)}\left(\exp -\frac{1}{2}A_t^{(q)}\right) \tag{2.1.3}$$

1) *For every $s \geq 0$ and $\Gamma_s \in b(\mathcal{F}_s)$:*
$W_{z,t}^{(2,q)}(\Gamma_s)$ admits a limit $W_{z,\infty}^{(2,q)}(\Gamma_s)$ as $t \to \infty$:

$$W_{z,t}^{(2,q)}(\Gamma_s) \underset{t\to\infty}{\longrightarrow} W_{z,\infty}^{(2,q)}(\Gamma_s) \tag{2.1.4}$$

2) *$W_{z,\infty}^{(2,q)}$ is a probability on $(\Omega, \mathcal{F}_\infty)$ such that :*

$$W_{z,\infty}^{(2,q)}|_{\mathcal{F}_s} = M_s^{(2,q)} \cdot W_z^{(2)}|_{\mathcal{F}_s}$$

where $(M_s^{(2,q)},\ s \geq 0)$ is the $\big((\mathcal{F}_s,\ s\geq 0),\ W_z^{(2)}\big)$ martingale defined by :

$$M_s^{(2,q)} = \frac{\varphi_q(X_s)}{\varphi_q(z)}\ \exp\left(-\frac{1}{2}A_s^{(q)}\right) \tag{2.1.5}$$

3) *The function $\varphi_q : \mathbb{C} \to \mathbb{R}_+$ featured in (2.1.5) is strictly positive, continuous and satisfies :*

$$\varphi_q(z) \underset{|z|\to\infty}{\sim} \frac{1}{\pi}\ \log\big(|z|\big) \tag{2.1.6}$$

It may be defined via one or the other of the following descriptions :
i) φ_q is the unique solution of the Sturm-Liouville equation :

$$\Delta\varphi = q\cdot\varphi \quad \textit{(in the sense of Schwartz distributions)}$$