Chapter 1. On a remarkable σ-finite measure $\mathbf{W}$ on path space, which rules penalisations for linear Brownian motion

1.0 Introduction.

1.0.1 $\big(\Omega, (X_t, \mathcal{F}_t),\ t \geq 0, \mathcal{F}_\infty, W_x(x \in \mathbb{R})\big)$ denotes the canonical realisation of 1-dimensional Brownian motion. $\Omega = \mathcal{C}(\mathbb{R}_+ \to \mathbb{R})$, $(X_t, t \geq 0)$ is the coordinate process on this space and $(\mathcal{F}_t,\ t \geq 0)$ denotes its natural filtration ; $\mathcal{F}_\infty = \underset{t\geq 0}{\vee} \mathcal{F}_t$. For every $x \in \mathbb{R}$, W_x denotes Wiener measure on $(\Omega, \mathcal{F}_\infty)$ such that $W_x(X_0 = x) = 1$. We write W for W_0 and if Z is a r.v. defined on $(\Omega, \mathcal{F}_\infty)$, we write $W_x(Z)$ for the expectation of Z under the probability W_x.

1.0.2 In a series of papers $\big([\text{RVY}, i],\ i = I, II, \cdots, X\big)$ we have studied various penalisations of Wiener measure with certain positive functionals $(F_t,\ t \geq 0)$; that is for each functional $(F_t,\ t \geq 0)$ in a certain class, we have been able to show the existence of a probability W_∞^F on $(\Omega, \mathcal{F}_\infty)$ such that : for every $s \geq 0$ and every $\Gamma_s \in b(\mathcal{F}_s)$, the space of bounded $\mathcal{F}_s$ measurable variables :

$$\lim_{t\to\infty} \frac{W(\Gamma_s F_t)}{W(F_t)} = W_\infty^F(\Gamma_s) \tag{1.0.1}$$

In this paper, we shall construct a positive and σ-finite measure $\mathbf{W}$ on $(\Omega, \mathcal{F}_\infty)$ which, in some sense, "rules all these penalisations jointly".

1.0.3 In Section 1.1 of this chapter, we show the existence of $\mathbf{W}$ and we describe some of its properties.
In Section 1.2, we show how to associate to $\mathbf{W}$ a family of $\big((\mathcal{F}_t,\ t \geq 0), W\big)$ martingales $\big(M_t(F),\ t \geq 0\big)$ $\big(F \in L^1_+(\mathcal{F}_\infty, \mathbf{W})\big)$. We study the properties of these martingales and give many examples.
In Section 1.3, we describe links between $\mathbf{W}$ and a σ-finite measure $\mathbf{\Lambda}$ which is defined as the "law" of the total local time of the canonical process under $\mathbf{W}$ in Chapter 3 of [RY, M]. In particular, we construct an invariant measure $\widetilde{\mathbf{\Lambda}}$ for the Markov process $\big((X_t, L_t^\bullet),\ t \geq 0\big)$ (and $\widetilde{\mathbf{\Lambda}}$ is intimately related to $\mathbf{\Lambda}$). Here, $L_t^\bullet$ denotes the local times process $(L_t^x, x \in \mathbb{R}_+)$, so that this Markov process $(X, L^\bullet)$ takes values in $\mathbb{R} \times \mathcal{C}(\mathbb{R} \longrightarrow \mathbb{R}_+)$.

1.0.4 Notations : As certain σ-finite measures play a prominent role in our paper, we write them, as a rule, in bold characters. Thus, no confusion should arise between the σ-finite measure $\mathbf{W}_x$ and the Wiener measure W_x.

1.1 Existence of W and first properties.

Our aim in this section is to define, via Feynman-Kac type penalisations, a positive and σ-finite measure $\mathbf{W}$ on $(\Omega, \mathcal{F}_\infty)$. Moreover, independently from this penalisation procedure, we give several remarkable descriptions of $\mathbf{W}$.

1.1.1 A few more notations.
$\big(\Omega, (X_t, \mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty, W_x(x \in \mathbb{R})\big)$ denotes the canonical realisation of 1-dimensional Brownian motion.
We denote by $\mathcal{I}$ the set of positive Radon measures $q(dx)$ on $\mathbb{R}$, such that :

$$0 < \int_0^\infty \big(1 + |x|\big)\ q(dx) < \infty \tag{1.1.1}$$

For every $q \in \mathcal{I}$, $(A_t^{(q)},\ t \geq 0)$ denotes the additive functional defined by :

$$A_t^{(q)} := \int_{\mathbb{R}} L_t^y\ q(dy) \tag{1.1.2}$$