Supplements.

§1. Borel's density theorem for fuchsian groups.[1] Let $G_{\mathbf{R}} = PSL_2(\mathbf{R})$. Then all finite dimensional irreducible ordinary representations of $G_{\mathbf{R}}$ are given by ρ_n ($n = 0, 1, 2, \cdots$), defined in Chapter 3, §3. Since they are algebraic representations, it is clear that if Δ is a subgroup of $G_{\mathbf{R}}$ not contained in any proper algebraic subgroup of $G_{\mathbf{R}}$, then $\rho_n|\Delta$ is also irreducible. In particular, if Δ is a discrete subgroup of $G_{\mathbf{R}}$ whose quotient has finite invariant volume, then Δ is Zariski dense in $G_{\mathbf{R}}$ (a special case of Borel's density theorem [**1**]; but since $\dim G_{\mathbf{R}} = 3$ is small, it can also be checked directly); hence $\rho_n|\Delta$ is irreducible.

In particular, $\rho_1|\Delta$ is irreducible. Since ρ_1 is equivalent to the adjoint representation Ad. of $G_{\mathbf{R}}$ in its Lie algebra $\mathfrak{g}_{\mathbf{R}}$, this shows that no proper Lie subalgebra $\neq \{0\}$ of $\mathfrak{g}_{\mathbf{R}}$ is invariant by $\mathrm{Ad}\,\Delta$. Now if $H_{\mathbf{R}}$ is a closed subgroup of $G_{\mathbf{R}}$ containing Δ with $(H_{\mathbf{R}} : \Delta) = \infty$, then $H_{\mathbf{R}}$ is non-discrete, and hence the corresponding Lie subalgebra $\mathfrak{h}_{\mathbf{R}}$ is non-trivial. But $\mathfrak{h}_{\mathbf{R}}$ is invariant by $\mathrm{Ad}\,H_{\mathbf{R}}$, and hence also by $\mathrm{Ad}\,\Delta$. Therefore $\mathfrak{h}_{\mathbf{R}} = \mathfrak{g}_{\mathbf{R}}$; hence $H_{\mathbf{R}} = G_{\mathbf{R}}$ (since $G_{\mathbf{R}}$ is connected).

Therefore, if $\tilde{\Delta}$ is a group with $G_{\mathbf{R}} \supset \tilde{\Delta} \supset \Delta$ and with $(\tilde{\Delta} : \Delta) = \infty$, then $\tilde{\Delta}$ is dense in $G_{\mathbf{R}}$.

Supplements to Chapter 1.

§2. A generalization of Lemma 10 of Chapter 1. Here, we shall verify the following assertion.[2]

The Lemma 10 of Chapter 1 remains valid if we weaken the compactness assumption of the quotient $G_{\mathbf{R}}/\Delta$ and replace it by the finiteness of volume, and if we assume that $f(z)$ is a cusp form. (*also by Kuga.*)

Proof. As in §21 (Chapter 1), put

$$F(g) = f(g(\sqrt{-1})) \cdot j(g, \sqrt{-1}) \quad (g \in G_{\mathbf{R}}), \tag{1}$$

so that $F(g)$ is a Δ-invariant continuous function on $G_{\mathbf{R}}$. In this case, the quotient $G_{\mathbf{R}}/\Delta$ may not be compact, but we shall check that $|F(g)|$ still achieves its maximum value on

[1] This is referred to in the following places: Chapter 2, §7, §24, Chapter 3, §1, §8.

[2] This is used in the proofs of Theorem 7 (Chapter 1, Part 2) (the inequality (171)), and the Theorem in Supplement §6.