Part 2. Non-abelian classfields attached to subgroups of $\Gamma = PSL_2(\mathbf{Z}^{(p)})$ with finite indices.

Put $\Gamma = PSL_2(\mathbf{Z}^{(p)})$ and $\Gamma^* = \{x \in GL_2(\mathbf{Z}^{(p)}) | \det x \in \Pi\} / \pm \Pi$, where $\mathbf{Z}^{(p)} = \Pi \cdot \mathbf{Z}$ and $\Pi = p^{\mathbf{Z}}$ (the infinite cyclic group generated by p), so that $\Gamma^* \supset \Gamma$, $(\Gamma^* : \Gamma) = 2$. Our main purpose in Part 2 of this chapter is to show that the group Γ^* (resp. Γ, or other related groups) describes a certain "non-abelian classfield theory" over the rational function field $K^* = \mathbf{F}_p(\bar{j})$ (resp. $\mathbf{F}_{p^2}(\bar{j})$, or other related algebraic function fields). Namely, for each normal subgroup Γ' of (say) Γ^* with finite index, a finite Galois extension K' of K^* called the Γ'-*classfield* is defined, and the following main theorems are proved:

(i) for each Γ', the Γ'-classfield exists and is unique;
(ii) there is a certain isomorphism $\iota_{\Gamma'} : G(K'/K^*) \cong \Gamma^*/\Gamma'$;
(iii) the law of decomposition of prime divisors of K^* in K' is completely described by the primitive elliptic conjugacy classes of Γ^* (and the isomorphism $\iota_{\Gamma'}$).

More precisely, let $\wp(\Gamma^*)$, $\wp(K^*)$ and the bijection $\mathcal{J}^* : \wp(\Gamma^*) \to \wp(K^*)$ be as in §10 (Part 1), $\mathcal{J}^*$ being defined with respect to a fixed prime factor $\mathfrak{p}$ of p in $\mathbf{Q}^{\mathrm{a}}$ (the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$). Then a finite Galois extension K' over K is called a Γ'-classfield if the following condition (♯) (§29) is satisfied:

(♯) An ordinary prime divisor $\mathfrak{P}^0$ of K^* (i.e., those $\mathfrak{P}^0$ contained in $\wp(K^*)$) is decomposed completely [5] in K' if and only if Γ_z^* is contained in Γ'; where $z (\in \mathfrak{H})$ is a representative of the Γ^*-equivalence class $\mathcal{J}^{*-1}(\mathfrak{P}^0)$, and Γ_z^* denotes its stabilizer in Γ^*.

With this definition, we have the following main theorems (§30):

Main Theorem (Γ^*-1). *For each Γ', Γ'-classfield exists and is unique.*

Main Theorem (Γ^*-2). *Let $\mathfrak{K}$ be the composite of all Γ'-classfields, where Γ' runs over all normal subgroups of Γ^* with finite indices. Then there is a dense injection $\iota : \Gamma^* \to G(\mathfrak{K}/K^*)$ satisfying the following conditions:*

(i) *ι induces an isomorphism of the completion of Γ^* with respect to "subgroups with finite indices topology" and $G(\mathfrak{K}/K^*)$; hence subgroups of Γ^* with finite indices and finite extensions of K^* contained in $\mathfrak{K}$ correspond in a one-to-one manner. Moreover, if Γ' is any normal subgroup of Γ^* with finite index, then the corresponding finite extension of K^* is nothing but the Γ'-classfield.*
(ii) *Let $\mathfrak{P}^0$ be any ordinary prime divisor of K^*, let z be a representative of $\mathcal{J}^{*-1}(\mathfrak{P}^0)$, and let Γ_z^* be the stabilizer of z in Γ^*. Let E_z^* be the torsion subgroup of Γ_z^*, and let γ be a positive generator of Γ_z^** mod *E_z^* with respect to $\mathfrak{p}$ (see §23). Then $\mathfrak{P}^0$ has an extension $\mathfrak{P}_z$ to $\mathfrak{K}$ whose inertia group is $\iota(E_z^*)$ and whose Frobenius substitution is $\iota(\gamma)$ (mod $\iota(E_z^*)$).*

[5] i.e., the relative degree is one.