CHAPTER 4

Part 1. Examples of Γ.

In Part 1 of this chapter, we shall give some examples of Γ. They are obtained from quaternion algebras A over totally real algebraic number fields F; and up to commensurability, they are the only examples of Γ that we know at present. We shall also prove that if L is a quasi-irreducible $G_\mathfrak{p}$-field over **C** such that the corresponding discrete subgroup is commensurable with one obtained from a quaternion algebra A over F, then the field k_0 (defined by Theorem 5 of Chapter 2) contains F (see Theorem 1, §5).

Examples of Γ.

§1. Quaternion algebra. By a quaternion algebra over a field F, we mean a simple algebra A with center F and with $[A : F] = 4$. The simplest example is $A = M_2(F)$, and all other quaternion algebras are division algebras. In the following, we shall make no distinction between two quaternion algebras over F which are isomorphic over F. If F is algebraically closed (e.g., if $F = \mathbf{C}$), then $A = M_2(F)$ is the only quaternion algebra over F. If $F = \mathbf{R}$ or $F = k_\mathfrak{p}$ ($\mathfrak{p}$-adic number field), then there is a *unique* division quaternion algebra over F, which will be denoted by $D_\mathbf{R}$ or $D_\mathfrak{p}$ respectively.

Now let F be an algebraic number field, and let $\mathfrak{p}$ be a prime divisor (finite or infinite) of F. Denote by $F_\mathfrak{p}$ the $\mathfrak{p}$-adic completion of F, so that either $F_\mathfrak{p} \cong \mathbf{C}$, or $F_\mathfrak{p} \cong \mathbf{R}$, or $F_\mathfrak{p}$ is a $\mathfrak{p}$-adic number field. For each quaternion algebra A over F, put $A_\mathfrak{p} = A \otimes_F F_\mathfrak{p}$; hence $A_\mathfrak{p}$ is a quaternion algebra over $F_\mathfrak{p}$. Therefore, if $F_\mathfrak{p} \cong \mathbf{C}$, $A_\mathfrak{p}$ must be $M_2(\mathbf{C})$, and if $F_\mathfrak{p} \not\cong \mathbf{C}$, then there are two possibilities for $A_\mathfrak{p}$; namely, $M_2(F_\mathfrak{p})$ or $D_\mathfrak{p}$ (or $D_\mathbf{R}$ if $F_\mathfrak{p} \cong \mathbf{R}$). A prime divisor $\mathfrak{p}$ of F is called *unramified* in A if $A_\mathfrak{p} \cong M_2(F_\mathfrak{p})$ holds, and *ramified* if $A_\mathfrak{p} \not\cong M_2(F_\mathfrak{p})$. Denote by $\delta(A)$ the set of all prime divisors of F which are ramified in A. Then it is well-known that $\delta(A)$ is finite and that its cardinal number is even. Conversely, if δ is any finite set of prime divisors of F not containing complex prime divisors and having even cardinal number, then there exists a quaternion algebra A over F, unique up to an isomorphism over F, such that $\delta = \delta(A)$;

$$(1) \qquad A \underset{1:1}{\leftrightarrow} \delta.$$